Overview and Comparison of Localic and Fixed-Basis Topological Products

Fatma Bayoumi[1]
Department of Mathematics, Faculty of Science
Benha University, Benha 13518, Egypt
fatma_bayoumi@hotmail.com

Stephen E. Rodabaugh[2]
Institute for Applied Topology and Topological Structures
College of Science, Technology, Engineering, Mathematics (STEM)
Youngstown State University, Youngstown, OH, USA 44555-3347
rodabaug@math.ysu.edu

Abstract

This paper studies localic products, traditional topological products, and L-topological products, and gives a complete outline of the localic product. Comparisons of localic and L-topological products are absent in the literature, and this paper answers longstanding open questions in that area as well as provides a complete proof of the classical comparison theorem for localic and traditional topological products. This paper contributes several L-valued comparison theorems, one of which states: the localic and L-topological products of L-topologies are order isomorphic if and only if the localic product is L-spatial, providing L is itself spatial and the family of L-topological spaces is "prime separated". These last two conditions always hold in the traditional setting, capturing the traditional comparison theorem as a special case, and the prime separation condition is satisfied by important lattice-valued examples that include the fuzzy real line and the fuzzy unit interval for L any complete Boolean algebra, and the alternative fuzzy real line and fuzzy unit interval for L any (semi)frame. Separation conditions help control the "sloppy" behavior of the L-topological product when $|L| > 2$, and several separation conditions are studied in this context; and it should be noted that localic products have a point-free version of the "product" separation condition considered in this paper. The traditional comparison theorem is carefully proved both to fill gaps in the extant literature and to motivate the L-valued comparison theorem quoted above and reveal the special role played by cross sums of prime (L-)open subsets. En route, characterizations are given of prime L-open subsets of certain L-products, which in turn yield characterizations of prime open and irreducible closed subsets of traditional product spaces.
Keywords: Localic/topological products; cross products/sums of L-open subsets; frame/meetsemilattice coproducts; frame/meetsemilattice quotients; L-spatiality; L-spectra; prime L-open sets; normalized/conormalized/hypernormalized L-topological spaces; product/sum/join/prime/projection separated families of L-topological spaces.

1 Introduction and Extended Overview

Products play a crucial role in point-set topology and increasingly a correspondingly crucial role in point-set lattice-theoretic (poslat) topology. In 1976, Dowker & Papert [2] constructed the coproduct of frames, a deep construction which, based on the notion of a frame, furnishes the product for point-free or localic topology, a product packaged by Johnstone [9] in 1982 using sites and coverages, and then repackaged recently and quite accessibly by Pultr [13] using quotients of frames by binary relations. Other important papers include [**?**] and [12], in the latter of which localic products play a special role in topological games and the product of strongly Baire topological spaces. It should be mentioned that localic products are critical to the completeness of the category **TopSys** of topological systems [28] important in semantic domains and to the completeness of the category **Loc-Top** of variable-basis topological spaces [18, 19, 23] important in fuzzy sets and to which **TopSys** is fundamentally related [1, 27].

Unless stated otherwise, L in the sequel is a frame.

[1]Support of a sabbatical at Youngstown State University during the 2007–2008 academic year by Benha University and the government of the Arab Republic of Egypt is gratefully acknowledged.
[2]Corresponding author.

1.1 Question. The fundamental question is the following: how does the notion of localic product compare with corresponding notions for traditional and lattice-valued topology? This question can take various forms, in each of which the localic product is denoted by $\bigoplus$ and the product topology by $\bigotimes$:

1. Given a family $\{(X_\gamma, \mathfrak{T}_\gamma) : \gamma \in \Gamma\} \subset |\mathbf{Top}|$, the corresponding product space $\left(\prod_{\gamma\in\Gamma} X_\gamma, \bigotimes_{\gamma\in\Gamma} \mathfrak{T}_\gamma\right)$, and the localic product $\bigoplus_{\gamma\in\Gamma} \mathfrak{T}_\gamma$, how do $\bigotimes_{\gamma\in\Gamma} \mathfrak{T}_\gamma$ and $\bigoplus_{\gamma\in\Gamma} \mathfrak{T}_\gamma$ compare as frames; e.g., are they necessarily order-isomorphic? See [2, 9]
2. Given a family $\{(X_\gamma, \tau_\gamma) : \gamma \in \Gamma\} \subset |L\text{-}\mathbf{Top}|$, the corresponding L-topological product space $\left(\prod_{\gamma\in\Gamma} X_\gamma, \bigotimes_{\gamma\in\Gamma} \tau_\gamma\right)$, and the localic product $\bigoplus_{\gamma\in\Gamma} \tau_\gamma$, how do $\bigotimes_{\gamma\in\Gamma} \tau_\gamma$ and $\bigoplus_{\gamma\in\Gamma} \tau_\gamma$ compare as frames; e.g., are they necessarily order-isomorphic? See [18, 19, 21].
3. Given a family $\{(X_\gamma, L_\gamma, \tau_\gamma) : \gamma \in \Gamma\} \subset |\mathbf{Loc\text{-}Top}|$ [23], the corresponding product space $\left(\prod_{\gamma\in\Gamma} X_\gamma, \bigoplus_{\gamma\in\Gamma} L_\gamma, \bigotimes_{\gamma\in\Gamma} \tau_\gamma\right)$, and the localic product $\bigoplus_{\gamma\in\Gamma} \tau_\gamma$, how do $\bigotimes_{\gamma\in\Gamma} \tau_\gamma$ and $\bigoplus_{\gamma\in\Gamma} \tau_\gamma$ compare as frames; e.g., are they necessarily order-isomorphic?

Question 1.1(1, 2) concerns *fixed-basis* topology and is the primary focus of this paper, while Question 1(3) concerns a form of *variable-basis* topology and is the subject of future work, as is also the investigation of these product questions when the underlying lattices of membership values are allowed to be quantales or other extensions of frames. We note that preliminary discussions on Question 1(2) are given in [18, 19, 21] and the present work may be viewed as a long overdue, extensive update and expansion of those discussions.

1.2 Theorem (comparison of localic and traditional topological products). $\bigoplus_{\gamma\in\Gamma} \mathfrak{T}_\gamma \cong \bigotimes_{\gamma\in\Gamma} \mathfrak{T}_\gamma$ if and only if $\bigoplus_{\gamma\in\Gamma} \mathfrak{T}_\gamma$ is spatial.

1.3 Proposal (comparison of localic and L-topological products). $\bigoplus_{\gamma\in\Gamma} \tau_\gamma \cong \bigotimes_{\gamma\in\Gamma} \tau_\gamma$ if and only if $\bigoplus_{\gamma\in\Gamma} \tau_\gamma$ is L-spatial.

The statement of Theorem 1.2 appears in [2] essentially without proof. Its proof is decidedly nontrivial (in fact, rather deep) and there are only very partial proofs in the literature. This paper gives complete proofs (more than one) of Theorem 1.2 (Subsections 7.3, 7.4). Proposal 1.3 is stated in essentially this form in [18]; and it is not known whether Proposal 1.3 in full generality is true, though necessity is always true:

1.4 Theorem [18, 19, 21]. Necessity in Proposal 1.3 always holds.

The following alternative to Proposal 1.3 is proved in Section 4.

1.5 Theorem. $\bigoplus_{\gamma\in\Gamma} \tau_\gamma \cong \bigotimes_{\gamma\in\Gamma} (\Phi_L)^{\rightarrow}(\tau_\gamma)$ if and only if $\bigoplus_{\gamma\in\Gamma} \tau_\gamma$ is L-spatial (where $(\Phi_L)^{\rightarrow}(\tau_\gamma)$ is the L-topology for the L-spectrum τ_γ—see Section 4 below). More generally, $\forall M$ a subframe of L such that L is M-spatial, $\bigoplus_{\gamma\in\Gamma} \tau_\gamma \cong \bigotimes_{\gamma\in\Gamma} (\Phi_M)^{\rightarrow}(\tau_\gamma)$ if and only if $\bigoplus_{\gamma\in\Gamma} \tau_\gamma$ is L-spatial, where $\bigotimes_{\gamma\in\Gamma} (\Phi_M)^{\rightarrow}(\tau_\gamma)$ is the L-topological product (in L-**Top**).

Sufficiency in Proposal 1.3 is true—Proposal 1.3 then becomes a theorem—under the antecedents of the next five theorems contributed by Sections 7, 8, and 9 of this paper. The first of these theorems includes Theorem 1.2 for $L = \mathbf{2}$ if the spaces in the latter are sober (and so, in particular, if they are Hausdorff) and generalizes the theorem proved in Paragraph II.2.13 of [9]; and the second theorem significantly generalizes the first by weakening the condition of L-sobriety. In the third and fifth theorems, Proposal 1.3 becomes a theorem under conditions each of which captures Theorem 1.2 when $L = \mathbf{2}$; and the fourth theorem is intermediate between the second and third theorems in its antecedent.

1.6 Theorem (cf. [18]). Let $\{(X_\gamma, \tau_\gamma) : \gamma \in \Gamma\}$ be a family of L-sober topological spaces (Definition 4.1 below). Then (sufficiency in) Proposal 1.3 holds.

1.6.1 Theorem (cf. [18]). Let $\{(X_\gamma, \tau_\gamma) : \gamma \in \Gamma\}$ be a family of L-S_0 topological spaces (Definition 4.1 below). Then (sufficiency in) Proposal 1.3 holds.

1.7 Theorem. Let L be spatial and $\{(X_\gamma, \tau_\gamma) : \gamma \in \Gamma\}$ be a prime separated family of L-topological spaces (AC if Γ nonfinite). Then (sufficiency in) Proposal 1.3 holds.

1.7.1 Theorem (cf. [18]). Let $\{(X_\gamma, \tau_\gamma) : \gamma \in \Gamma\}$ be a normalized and prime separated family of q-L-S_0 topological spaces (AC if Γ nonfinite). Then (sufficiency in) Proposal 1.3 holds.

1.8 Theorem. If $\{(X_\gamma, \tau_\gamma) : \gamma \in \Gamma\}$ is a join separated and product separated family of L-topological spaces, then (sufficiency in) Proposal 1.3 holds.

1.8.1 Remark. The properties of "product separation", "prime separation", "normalized" are technical conditions which always hold for traditional spaces. In the case of two traditional spaces $(X, \mathfrak{T}), (Y, \mathfrak{S})$, the first two conditions are the properties, respectively, that if $A \times B \neq \varnothing$, then

$$A \times B \subset C \times D \;\Leftrightarrow\; A \subset C \text{ and } B \subset D,$$

$$A \times B \subset C + D \;\Leftrightarrow\; A \subset C \text{ or } B \subset D,$$

where

$$C + D = (C \times Y) \cup (X \times D).$$

And "normalized" is the property that for each nonempty open subset U, say of $(X, \mathfrak{T})$, it is the case that

$$||\chi_U|| \equiv \bigvee_{x \in X} \chi_U(x) = \top.$$

The fuzzy analogues of the separation properties (see Section 5 below) do not generally hold for product L-powersets—products and sums of L-subsets are "messy" when $|L| \geq 3$, the factors and terms "spilling" into each other. But there are many important L-topological spaces which form families whose product L-powersets satisfy these properties: the fuzzy real lines and fuzzy unit intervals for L any complete Boolean algebra; the alternative fuzzy real lines and fuzzy unit intervals for L any semiframe; the L-soberifications of any **2**-topological spaces for any L a semiframe; and many other spaces. The motivation for the "join separation condition" assumed in 1.8 is more subtle and given in Section 9.

A weaker result than sufficiency in Proposal 1.3 is given by our last theorem:

1.9 Theorem. If $\bigoplus_{\gamma \in \Gamma} \tau_\gamma$ is L-spatial, then $\bigotimes_{\gamma \in \Gamma} \tau_\gamma$ is a sublocale of $\bigoplus_{\gamma \in \Gamma} \tau_\gamma$.

1.10 Organization of Paper. The rest of this paper is organized as follows: Section 2 gives needed notation and summarizes products in **Top** and L-**Top**; Section 3 gives a needed, detailed summary of the construction of products in **Loc** based on the elegant approach of [13]; notions of points and the $L\Omega \dashv LPT$ adjunction, important tools in this paper, are catalogued and used in Section 4 to prove Theorems 1.4, 1.5, 1.6; various separation conditions concerning products in L-**Top** and their relationships to powerset operators are developed in Section 5, including the crucial *prime separation condition*; Section 6 sets up Sections 7, 8, 9 and proves Theorem 1.6.1; Theorem 1.7 and Theorem 1.2 are proved in Section 7, providing a proof of Theorem 1.2 that is without gaps as well as characterizations of prime L-open subsets of certain L-topological product spaces and characterizations of prime open and irreducible closed subsets of traditional product spaces; Theorem 1.7.1 is proved in Section 8; and Theorems 1.8 and 1.9 are proved in Section 9.

2 Preliminary notions

2.1 Frames

Unless stated otherwise, L is a *frame*, that is, a complete lattice satisfying the first infinite distributive law or, equivalently, equipped with the Heyting implication operation $\rightarrow$, the latter defined by saying that

$$\forall a, b, c \in L,\; a \rightarrow b \geq c \Leftrightarrow a \wedge c \leq b.$$

The *inconsistent* frame **1** (with $\perp = \top$) is allowed; and the *crisp* frame $\mathbf{2} = \{\perp, \top\}$. The notion of *coframe* is dual to that of frame, namely, a complete lattice satisfying the second infinite distribute law; alternatively, it is the dual lattice of a frame. And a *diframe* is both a frame and a coframe, i.e., a complete lattice which is *weakly completely distributive*—the regular open subsets of $\mathbb{R}$ comprise a diframe which is not completely distributive; and we note that some workers (but not all) use *biframe* for such a structure.

The category **Frm** comprises all frames along with all mappings between frames which preserve arbitrary $\bigvee$ and finite $\wedge$, along with the composition and identities of **Set**. The dual or opposite category $\mathbf{Frm}^{op}$ is denoted **Loc**, and in the latter the objects are also called *locales*.

2.2 Powersets and powerset operators

Given a function $f : X \to Y$, the traditional and Zadeh L-valued (upper) image and preimage and lower image operators are defined respectively as follows:

$$\begin{array}{llll}
f^{\to} & : \wp(X) \to \wp(Y) & \text{by} & f^{\to}(A) = \{f(y) \in Y : x \in A\}, \\
f^{\leftarrow} & : \wp(X) \leftarrow \wp(Y) & \text{by} & f^{\leftarrow}(B) = \{x \in X : f(x) \in B\}, \\
f_{\to} & : \wp(X) \to \wp(Y) & \text{by} & f_{\to}(A) = \{y \in Y : f^{\leftarrow}\{y\} \subset A\}, \\
f_L^{\to} & : L^X \to L^Y & \text{by} & f_L^{\to}(a)(y) = \bigvee_{x \in f^{\leftarrow}\{y\}} a(x), \\
f_L^{\leftarrow} & : L^X \leftarrow L^Y & \text{by} & f_L^{\leftarrow}(b) = b \circ f, \\
f_{L\to} & : L^X \to L^Y & \text{by} & f_{L\to}(a)(y) = \bigwedge_{x \in f^{\leftarrow}\{y\}} a(x).
\end{array}$$

Note that L is often dropped from the notation when L is understood from the context. It is well-known that

$$f^{\to} \dashv f^{\leftarrow} \dashv f_{\to}, \quad f_L^{\to} \dashv f_L^{\leftarrow} \dashv f_{L\to}$$

and that the Zadeh operators are, respectively, the extensions of the traditional operators; e.g., $f_{\mathbf{2}}^{\to}(\chi_A) = \chi_{f^{\to}(A)}$.

2.2.1 Proposition. Let $f : X \to Y$ be a function.

1. $f^{\to}$ and $f_L^{\to}$ preserve arbitrary joins, $f^{\leftarrow}$ and $f_L^{\leftarrow}$ preserve both arbitrary joins and arbitrary meets, and $f_{L\to}$ and $f_{\to}$ preserve arbitrary meets.
2. The following are equivalent: f is bijective; $f_L^{\to}$ is bijective and $(f_L^{\to})^{-1} = f_L^{\leftarrow} = \left(f^{-1}\right)_L^{\to}$; $f_L^{\leftarrow}$ is bijective and $(f_L^{\leftarrow})^{-1} = f_L^{\to} = \left(f^{-1}\right)_L^{\leftarrow}$; and $f_L^{\to} = f_{L\to}$.
3. Let f be injective. Then $\forall a \in L^X$, $\forall \{a_\gamma\}_{\gamma\in\Gamma} \subset L^X$, $\forall y \in Y$, the following hold:

$$f_L^{\to}(a)(y) = \begin{cases} \bot, & y \notin f^{\to}(X) \\ a(x), & \exists!\, x \in X,\, y = f(x) \end{cases},$$

$$\left(f_L^{\to}(a)_{|\, f^{\to}(X)}\right)(y) = a(x), \text{ where } y = f(x),$$

$$f_L^{\to}\left(\bigwedge_{\gamma\in\Gamma} a_\gamma\right) = \bigwedge_{\gamma\in\Gamma} f_L^{\to}(a_\gamma).$$

2.2.2 Definition (cf. [26]). Let X be a set and $a \in L^X$. Then the *norm* $||a||$ and *conorm* $||a||_{co}$ of $a \in L^X$ are respectively given by

$$||a|| = \bigvee_{x\in X} a(x), \quad ||a||_{co} = \bigwedge_{x\in X} a(x);$$

and a is *normalized* [*conormalized*] if $||a|| = \top$ [$||a||_{co} = \bot$, respectively]. Further, a family $\mathfrak{A} \subset L^X$ of L-subsets is:

1. *normalized* if each $a \in \mathfrak{A} - \{\underline{\bot}\}$ is normalized;
2. *conormalized* if each $a \in \mathfrak{A} - \{\underline{\top}\}$ is conormalized;
3. *binormalized* if it is both normalized and conormalized;

4. *hypernormalized* if $\forall a, b \in \mathfrak{A}$ with $a \nleq b$, $\exists\, x \in X$, $a\,(x) = \top$, $b\,(x) = \bot$.

2.2.3 Proposition. Binormalized $\Rightarrow$ each of normalized and conormalized; if $\mathfrak{A}$ includes $\{\underline{\bot}, \underline{\top}\}$, then all of these conditions are consequences of hypernormalized; and none of these implications reverses.

Proof. Suppose $\mathfrak{A}$ includes $\{\underline{\bot}, \underline{\top}\}$ and let $a \in \mathfrak{A} \in \{\underline{\bot}\}$ be given. Assume $\mathfrak{A}$ is hypernormalized and choose $b = \underline{\bot}$. Then $a \nleq b$. So $\exists\, x \in X$, $a\,(x) = \top$, $b\,(x) = \bot$; and in particular, we have $||a|| = \top$. Hence hypernormalized implies normalized, and the proof that hypernormalized implies conormalized is dual. All the other assertions are trivial. $\square$

2.2.4 Proposition.

1. The Zadeh upper [lower] image of a normalized [co-normalized] L-subset is normalized [co-normalized].
2. If a ground mapping f is injective and $\mathfrak{A}$ is a hypernormalized family on the domain, then both $(f_L^{\rightarrow})^{\rightarrow}(\mathfrak{A})$ and $(f_{L\rightarrow})^{\rightarrow}(\mathfrak{A})$ are hypernormalized.
3. If a ground mapping f is surjective and $\mathfrak{A}$ is a hypernormalized family on the codomain, then $(f_L^{\leftarrow})^{\rightarrow}(\mathfrak{A})$ is hypernormalized.

For more details on the powerset monads for L-valued sets, see [25] and its references.

2.3 Subframes and sublocales

A is a *subframe* of frame B if $A \subset B$ and A is closed in B under arbitrary $\bigvee$ and finite $\wedge$.

For frame A, the mapping $\nu : A \to A$ is a *nucleus* if $\forall a, b \in A$ the following hold::

(N1) $a \leq \nu(a)$ (ν enlarges);

(N2) $\nu\,(\nu\,(a)) \leq \nu\,(a)$ (ν fixes outputs: $\nu\,(\nu\,(a)) = \nu\,(a)$);

(N3) $\nu\,(a \wedge b) = \nu\,(a) \wedge \nu\,(b)$ (ν preserves binary meets).

S is a *sublocale* of locale A if $S \subset A$ and $\exists$ a nucleus $\nu : A \to A$ with $\nu^{\rightarrow}(A) = S$. It is equivalent to say that S is the image of A under some frame morphism; and it is also equivalent to say that S satisfies:

(S1) S is closed in A under arbitrary $\bigwedge$, and

(S2) S is closed in A "under consequents", i.e., $\forall a \in A$, $\forall b \in S$, $a \to b \in S$.

If $\nu : A \to A$ is a nucleus, we sometimes abuse the terminology and also speak of $\nu : A \to \nu^{\rightarrow}(A)$ as a nucleus.

2.4 Topological notions

The category L-**Top** comprises all L-topological spaces—ordered pairs (X, τ) with X a set and τ a subframe of L^X, along with L-continuous mappings between them—$f : (X, \tau) \to (Y, \sigma)$ with $f : X \to Y$ satisfying $\forall v \in \sigma$, $f_L^{\leftarrow}(v) \in \tau$. The category **Top** of ordinary topological spaces and continuous maps between them is functorially isomorphic to **2-Top** (L-**Top** with $L = \mathbf{2}$) via the characteristic functor G_χ defined by

$$G_\chi\,(X, \mathfrak{T}) = (X, G_\chi\,(\mathfrak{T}))\,, \quad G_\chi\,(\mathfrak{T}) = \{\chi_U : U \in \mathfrak{T}\}\,, \quad G_\chi\,(f) = f.$$

Recall that for A a subset of X and $(X, \mathfrak{T})$ a topological space, $\mathfrak{T}_A \equiv \{U \cap A : U \in \mathfrak{T}\})$ is the subspace topology on A; and for (X, τ) and L-topological space, $\tau_A \equiv \left\{u_{|A} : u \in \tau\right\}$ is the L-subspace topology on A. Also, if $\mathfrak{S}$ is a subbasis for $(X, \mathfrak{T})$, we write $\mathfrak{T} = \langle\langle\mathfrak{S}\rangle\rangle$; σ is a *subbasis* [*basis*] for (X, τ) means that

$$\begin{aligned} \tau &= \bigcap \{\hat{\tau} : (X, \tau) \in |L\text{-}\mathbf{Top}| \text{ and } \sigma \subset \hat{\tau}\} \\ [\tau &= \left\{\bigvee \mathcal{B} : \mathcal{B} \subset \sigma\right\}], \end{aligned}$$

in which case we write $\tau = \langle\langle\sigma\rangle\rangle$ [$\tau = \langle\mathcal{B}\rangle$].

2.4.1 Definition. Let $(X, \tau)\,, (Y, \sigma) \in |L\text{-}\mathbf{Top}|$ and $f : X \to Y$ be a mapping.

1. f is L-subbasic continuous if $\sigma = \langle\langle\omega\rangle\rangle$ and $\forall v \in \omega,\ f_L^{\leftarrow}(v) \in \tau$.

2. f is *L-open* if $\forall u \in \tau,\ f_L^{\rightarrow}(u) \in \sigma$; and f is *relatively L-open* if $\forall u \in \tau,\ f_L^{\rightarrow}(u) \in \sigma_{f^{\rightarrow}(X)}$. If $\tau = \langle\langle\omega\rangle\rangle$, then the modifier "subbasic" is used if u in the preceding sentence is restricted to being in ω.

3. f is an *L-embedding* if f is a injection which is L-continuous and relatively L-open; and f is an *L-homeomorphism* if f is a bijection which is L-continuous and whose inverse is L-continuous.

Note that an L-embedding is an L-homeomorphism onto the L-subspace of the image, and an L-homeomorphism is a surjective L-embedding.

2.4.2 Lemma. Let $(X,\tau)\,,\,(Y,\sigma) \in |L\text{-}\mathbf{Top}|$ and $f : X \to Y$ be a mapping.

1. f is L-continuous $\Leftrightarrow$ f is L-subbasic continuous.

2. If f is injective, then f is relatively L-open $\Leftrightarrow$ f is relatively L-subbasic open.

3. If f is an L-homeomorphism, then $\tau \cong \sigma$ as frames.

4. If f is an L-embedding, then τ is a sublocale of σ.

Proof. (1) is well-known (e.g., [23]) and (2) is straightforward. Concerning (3), since $f : X \to Y$ is bijective, then $f_L^{\rightarrow} : L^X \to L^Y$ is a bijection by 2.2.1. Further, L-continuity and L-openness insure that $f_L^{\rightarrow} : \tau \to \sigma$ is bijective. It suffices to invoke that $f_L^{\rightarrow}$ preserves joins to conclude that $f_L^{\rightarrow} : \tau \to \sigma$ is a frame isomorphism. As for (4), assume $f : (X,\tau) \to (Y,\sigma)$ is an L-embedding, so that $f : (X,\tau) \to \left(f^{\rightarrow}(X)\,,\sigma_{f^{\rightarrow}(X)}\right)$ is an L-homeomorphism. By (3),

$$(f_L^{\leftarrow})_{|\,\sigma_{f^{\rightarrow}(X)}} : \tau \leftarrow \sigma_{f^{\rightarrow}(X)}$$

is a frame isomorphism and hence a surjection. Now let $u \in \tau$. Then there is $\hat{v} \in \sigma_{f^{\rightarrow}(X)}$, $f_L^{\leftarrow}(\hat{v}) = u$; and there is $v \in \sigma$, $\hat{v} = v_{|\,f^{\rightarrow}(X)}$. Hence for $x \in X$,

$$f_L^{\leftarrow}(v)(x) = v(f(x)) = v_{|\,f^{\rightarrow}(X)}(f(x)) = \hat{v}(f(x)) = f_L^{\leftarrow}(\hat{v})(x) = u(x)\,,$$

and therefore $(f_L^{\leftarrow})_{|\,\sigma} : \tau \leftarrow \sigma$ is a surjective frame morphism. Therefore, τ is a sublocale of σ by 2.3 abaove. $\square$

2.4.3 Definition (topological products). The family $\{(X_\gamma, \mathfrak{T}_\gamma) : \gamma \in \Gamma\} \subset |\mathbf{Top}|$ has the categorical product in $\mathbf{Top}$ given by

$$\left(\left(\prod_{\gamma\in\Gamma} X_\gamma, \bigotimes_{\gamma\in\Gamma} \tau_\gamma\right), \{\pi_\gamma\}_{\gamma\in\Gamma}\right),$$

where $\left(\prod_{\gamma\in\Gamma} X_\gamma, \{\pi_\gamma\}_{\gamma\in\Gamma}\right)$ is the categorical product in $\mathbf{Set}$ and

$$\bigotimes_{\gamma\in\Gamma} \mathfrak{T}_\gamma = \left\langle\left\langle\left\{\pi_\gamma^{\leftarrow}(V) : \gamma\in\Gamma,\, V \in \mathfrak{T}_\gamma\right\}\right\rangle\right\rangle;$$

and, similarly, $\{(X_\gamma, \tau_\gamma) : \gamma \in \Gamma\} \subset |L\text{-}\mathbf{Top}|$ has the categorical product in L-$\mathbf{Top}$ given by

$$\left(\left(\prod_{\gamma\in\Gamma} X_\gamma, \bigotimes_{\gamma\in\Gamma} \tau_\gamma\right), \{\pi_\gamma\}_{\gamma\in\Gamma}\right),$$

where

$$\bigotimes_{\gamma\in\Gamma} \tau_\gamma = \left\langle\left\langle\left\{\pi_\gamma^{\leftarrow}(V) : \gamma\in\Gamma,\, V \in \tau_\gamma\right\}\right\rangle\right\rangle.$$

In both cases, double rhombic brackets indicate a subbasis as defined above.

Because of the infinite distributive law, the product topology and product L-topology can respectively be generated from these subbases via bases in the usual way. In this context we now define the "cross product" and "cross sum" of L-subsets. Given two traditional sets X, Y and $A \in \wp(X)$, $B \in \wp(Y)$, it is the case that the cross product $A \times B$ and cross sum $A + B$ have the following reformulations using the projections:

$$A \times B = \pi_1^{\leftarrow}(A) \cap \pi_2^{\leftarrow}(B), \quad A + B = \pi_1^{\leftarrow}(A) \cup \pi_2^{\leftarrow}(B).$$

By analogy, for $a \in L^X$, $b \in L^Y$, we define (suppressing the L),

$$a \boxtimes b = \pi_1^{\leftarrow}(a) \wedge \pi_2^{\leftarrow}(b), \quad a \boxplus b = \pi_1^{\leftarrow}(a) \vee \pi_2^{\leftarrow}(b).$$

This is equivalent to saying that $a \boxtimes b$, $a \boxplus b : X \times Y \to L$ by $\forall (x, y) \in X \times Y$,

$$(a \boxtimes b)(x, y) = a(x) \wedge b(y), \quad (a \boxplus b)(x, y) = a(x) \vee b(y).$$

This leads to the following definition:

2.4.4 Definition (cross products and cross sums of L-subsets). Let $\{X_\gamma : \gamma \in \Gamma\}$ be a family of sets, let $B \subset \Gamma$, and let $\{a_\beta : \beta \in B\}$ be a family of L-subsets such that $a_\beta \in L^{X_\beta}$. The (L-) *cross product* $\boxtimes_{\beta \in B} a_\beta : \prod_{\gamma \in \Gamma} X_\gamma \to L$ is defined by

$$(\boxtimes_{\beta \in B} a_\beta) \langle x_\gamma \rangle_{\gamma \in \Gamma} = \bigwedge_{\beta \in B} a_\beta(x_\beta),$$

and the dual (L-) *cross sum* $\boxplus_{\beta \in B} a_\beta : \prod_{\gamma \in \Gamma} X_\gamma \to L$ is defined by

$$(\boxplus_{\beta \in B} a_\beta) \langle x_\gamma \rangle_{\gamma \in \Gamma} = \bigvee_{\gamma \in B} a_\beta(x_\beta).$$

Both the cross product and cross sum are used in this paper, and it is important to note that the cross product is intimately related to the upper image operators of the projections and the cross sum is (dually) intimately related to the lower image operators of the projections—see Section 5. For now, we note finite cross products comprise the standard basis of the L-product topology (assuming L is a frame).

2.4.5 Proposition. It holds that

$$\bigotimes_{\gamma \in \Gamma} \tau_\gamma = \langle \{ \boxtimes_{i=1}^n u_{\gamma_i} : n \in \mathbb{N},\ \gamma_i \in \Gamma,\ u_{\gamma_i} \in \tau_{\gamma_i} \} \rangle,$$

where the single rhombic brackets indicate$\{ \boxtimes_{i=1}^n u_{\gamma_i} : n \in \mathbb{N},\ \gamma_i \in \Gamma,\ u_{\gamma_i} \in \tau_{\gamma_i} \}$ is a basis for $\bigotimes_{\gamma \in \Gamma} \tau_\gamma$.

Both the usual projections and generalized projections are needed in the latter sections of the paper, the latter defined by the following:

2.4.3 Definition (generalized projections). Given a product set $\prod_{\gamma \in \Gamma} X_\gamma$, the family $\left\{ \pi_{\prod_{\lambda \in \Lambda} X_\lambda} : \Lambda \subset \Gamma \right\}$ of **(generalized) projections** is given by

$$\pi_{\prod_{\lambda \in \Lambda} X_\lambda} \langle x_\gamma \rangle_{\gamma \in \Gamma} = \langle x_\lambda \rangle_{\lambda \in \Lambda}.$$

If Λ is the singleton $\{\beta\}$, then we recover the usual projection $\pi_\beta : \prod_{\gamma \in \Gamma} X_\gamma \to X_\beta$.

3 Products in Loc

Based on the elegant approach of [13], this section gives a complete, detailed outline of the construction of localic products needed in subsequent sections and en route fills a small gap of [13] by inserting Step 2 (3.5 below) into this construction—see Discussion 3.8 below.

We first note that the construction of **Loc** products is equivalent to the construction of concrete **Frm** coproducts. Beginning with the definition of coproducts in a category, we outline two fundamental tools needed in the construction of **Frm** coproducts and then give this construction in four steps.

3.1 Definition (coproducts in a category). Let $\{A_\gamma\}_{\gamma \in \Gamma}$ be a collection of objects in a category $\mathcal{C}$. Then $\left(\bigoplus_{\gamma \in \Gamma} A_\gamma, \{\iota_\gamma\}_{\gamma s \Gamma} \right)$ is the *coproduct* of $\{A_\gamma\}_{\gamma \in \Gamma}$ in $\mathcal{C}$ if the following hold:

1. $\bigoplus_{\gamma\in\Gamma} A_\gamma \in |\mathcal{C}|$, and $\beta \in \Gamma$, $\iota_\beta : A_\beta \to \bigoplus_{\gamma\in\Gamma} A_\gamma$ is a $\mathcal{C}$ morphism.
2. $\forall \left(B, \{\kappa_\gamma\}_{\gamma\in\Gamma}\right)$ satisfying (1), $\exists!\, h : \bigoplus_{\gamma\in\Gamma} A_\gamma \to B$, $\forall \beta \in \Gamma$, $\kappa_\beta = h \circ \iota_\beta$.

3.2 Tool 1 (downset functor [13]**).** Let $\mathbf{SLat}_\perp(\wedge)$ be the category of all meet semilattices with $\perp$ together with all mappings preserving finite $\wedge$ and $\perp$, along with the usual composition and identity morphisms. Note that $\mathbf{Frm}$ includes into $\mathbf{SLat}_\perp(\wedge)$, which inclusion is denoted $\hookrightarrow\ : \mathbf{Frm} \to \mathbf{SLat}_\perp(\wedge)$. Now $\hookrightarrow$ has a left-adjoint $\mathcal{D}$, called the *downset functor*, whose construction we need. Put $\mathcal{D} : \mathbf{Frm} \leftarrow \mathbf{SLat}_\perp(\wedge)$ as follows:

$$\mathcal{D}(S) = \{\downarrow A : \varnothing \neq A \subset S\},$$

where

$$\downarrow A = \bigcup_{a\in A} \downarrow(a) \quad \text{and} \quad \downarrow(a) = \{x \in S : x \le a\}$$

and

$$\mathcal{D}(f : S \to T) : \mathcal{D}(S) \to \mathcal{D}(T) \quad \text{by} \quad \mathcal{D}(f)(\downarrow A) = \ \downarrow f^{\to}(\downarrow A).$$

3.2.1 Downset Functor Theorem [13]. The following hold:

1. $\mathcal{D}(S)$ is a frame with bottom element $\{\perp\}$ and top element S.
2. $\mathcal{D}(f)$ is a frame morphism and $\mathcal{D}$ is a functor.
3. $\mathcal{D} \dashv\ \hookrightarrow$ with unit $\eta_S : S \to \mathcal{D}(S)$ in $\mathbf{SLat}_\perp(\wedge)$ given by $\eta_S(a) = \ \downarrow(a)$. For each $\mathbf{SLat}_\perp(\wedge)$ morphism $f : S \to C$ with C a frame, the unique factoring map $\overline{f} : \mathcal{D}(S) \to C$ satisfying
$$f = \overline{f} \circ \eta_S$$
is given by
$$\overline{f}(B) = \bigvee_{b\,\in\, B} f(b).$$
4. The unit η_S order-embeds S as a *generating set* of $\mathcal{D}(S)$. More precisely:
   (a) η_S is an order-isomorphism from S to $\eta_S^{\to}(S)$.
   (b) Each member of $\mathcal{D}(S)$ is a union of principal ideals from $\eta_S^{\to}(S)$, written $\mathcal{D}(S) = \langle \eta_S^{\to}(S)\rangle$.
5. The unit η_S is *epimorphic with respect to composition with frame maps following* η_S in the following sense: given frame maps $f, g : \mathcal{D}(S) \rightrightarrows B$ with $f \circ \eta_S = g \circ \eta_S$, it is the case that $f = g$.

We note that the proof of 3.2.1 requires the bottom element for the meet semilattices in questions *and* its preservation, i.e., that we start with the category $\mathbf{SLat}_\perp(\wedge)$ and not with the category $\mathbf{SLat}(\wedge)$—the category of all meet semilattices together with all mappings preserving finite $\wedge$. This is related to the insertion of Step 2 below into the construction of [13]; we also note that 3.2.1(5) is critical to verifying the couniversal property for the injections of the $\mathbf{Frm}$ coproduct at the conclusion of Step 4 (3.7) below; and see Discussion 3.8.

3.3 Tool 2 (quotient frames from binary relations [13]**).** Let A be a frame with implication operator $\to$, and let $R \subset A \times A$ be any binary operation on A. An element $s \in A$ is *(R-) saturated* if

$$\forall a, b, c \in A,\ a\, R\, b \ \Rightarrow\ (a \wedge c \le s \Leftrightarrow b \wedge c \le s).$$

The concept of saturated elements is intimately linked to the Heyting implication on A, and this implication and its properties are critical to the proofs of what follows. Put

$$A\,/\,R \equiv \{s \in A : s \text{ is saturated}\}.$$

Then $A\,/\,R$ is closed under arbitrary $\bigwedge$ of A; and $A\,/\,R$ is a frame (but need not be a subframe of A), called the *quotient frame of* A *(by* R*)*. Now put

$$\nu_R : A \to A\,/\,R$$

by

$$\nu_R(a) = \bigwedge \{s \in A / R : a \leq s\}.$$

If the relation R is understood, we may suppress the subscript.

3.3.1 Frame Quotient Theorem [13]. The following hold:

1. $\nu_R : A \to A / R \subset A$ is a nucleus, in fact an extremal epimorphism in **Frm**, and A / R is a sublocale of A.

2. $\forall a, b \in A,\ a\, R\, b \Rightarrow \nu_R(a) = \nu_R(b)$.

3. $\forall h : A \to B$ in **Frm** satisfying $(\forall a, b \in A,\ a\, R\, b \Rightarrow h(a) = h(b))$, $\exists!\, \overline{h} : A / R \to B$,
$$\overline{h} \circ \nu = h.$$

4. $\forall s \in A / R,\ \overline{h}(s) = h(s)$.

It should be noted that the relation R defined by $a\, R\, b \Leftrightarrow b = \neg\neg a$, where $\neg a = a \to \bot$, may be put on any frame A; the resulting quotient A / R is a complete Boolean algebra, called the *Booleanization B_A of A*; and each complete Boolean algebra arises in this way.

Given $\{A_\gamma : \gamma \in \Gamma\} \subset |\mathbf{Frm}|$, we are now in a position to construct its frame coproduct in four main steps. The main intuition is that we try to construct the traditional product topology without using any underlying carrier sets, that is, using only the topologies themselves. This is done by first imitating the construction of the basis of the product topology, and then trying to close up that basis with respect to arbitrary joins (or unions). The first two steps of the following construction are analogous to constructing the basis of the traditional product topology, while the last two steps focus on the much deeper question of closing up the "basis" with respect to arbitrary joins.

3.4 Step 1: Coproduct of $\{A_\gamma\}_{\gamma \in \Gamma}$ in SLat($\wedge$). Form the object $\prod_{\gamma\in\Gamma} A_\gamma$ of the product frame (constructed using the point-wise order)—we do *not* use the projections associated with this product, only the object. Now in **SLat**($\wedge$)—the category of meet semilattices with finite meet preserving mappings—make the following constructions:

$$\coprod_{\gamma\in\Gamma} A_\gamma = \left\{ \langle a_\gamma \rangle_{\gamma\in\Gamma} : a_\gamma = \top_\gamma \text{ for all but finitely many } \gamma \right\} \cup \left\{ \langle \bot_\gamma \rangle_{\gamma\in\Gamma} \right\},$$

$$\varphi_\beta : A_\beta \to \coprod_{\gamma\in\Gamma} A_\gamma \quad \text{by} \quad (\varphi_\beta(a))_\gamma = \begin{cases} a, & \beta = \gamma \\ \top_\gamma, & \beta \neq \gamma \end{cases}.$$

3.4.1 Theorem. The following hold:

1. $\coprod_{\gamma\in\Gamma} A_\gamma$ with the relative order is sub-meet-semilattice of $\prod_{\gamma\in\Gamma} A_\gamma$.

2. $\left(\coprod_{\gamma\in\Gamma} A_\gamma, \{\varphi_\gamma\}_{\gamma\in\Gamma}\right)$ is the coproduct of $\{A_\gamma : \gamma \in \Gamma\}$ in **SLat**($\wedge$).

3.4.2 Corollary. $\{\varphi_\gamma\}_{\gamma\in\Gamma}$ is collection-wise epimorphic (or an epi-sink) in **SLat**($\wedge$).

Corollary 3.4.2 is crucial to the verification of the **Frm** coproduct being constructed, in particular, the couniversal character of the injections of the **Frm** coproduct in Step 4 (3.7) below.

At this point of the construction, we already note a significant discrepancy between $\coprod_{\gamma\in\Gamma} A_\gamma$ and the basis of the traditional product topology: the bottom of the traditional basis is the empty set represented by any basic open set with the empty set in at least one factor, while the bottom of $\coprod_{\gamma\in\Gamma} A_\gamma$ has the bottom element of $\prod_{\gamma\in\Gamma} A_\gamma$ uniquely represented by that tuple which is bottom in every coordinate. More importantly, this behavior of $\coprod_{\gamma\in\Gamma} A_\gamma$ prevents the injections $\{\varphi_\gamma\}_{\gamma\in\Gamma}$ from preserving bottom elements (if $|\Gamma| \geq 2$). This deficiency of $\left(\coprod_{\gamma\in\Gamma} A_\gamma, \{\varphi_\gamma\}_{\gamma\in\Gamma}\right)$ is now addressed in Step 2 (3.5) *before* applying Tool 1 (3.2) and Tool 2 (3.3). The insertion of Step 2 into the construction at this point is essential and relates to the cboice of relation R in Step 4 (3.7) and, even more importantly, to the eventual couniversality of the injections of the frame coproduct at the conclusion of Step 4: both issues are discussed more fully in 3.8 below,

3.5 Step 2: Coproduct of $\{A_\gamma\}_{\gamma\in\Gamma}$ in $\mathbf{SLat}_\perp(\wedge)$. Put a binary relation Q on $\coprod_{\gamma\in\Gamma} A_\gamma$ from Step 1 as follows:

Case 1. $\forall\gamma\in\Gamma$, $a_\gamma\neq\perp$. In this case,

$$\langle a_\gamma\rangle_{\gamma\in\Gamma}\; Q\; \langle b_\gamma\rangle_{\gamma\in\Gamma} \;\Leftrightarrow\; \langle a_\gamma\rangle_{\gamma\in\Gamma} = \langle b_\gamma\rangle_{\gamma\in\Gamma}\,.$$

Case 2. $\exists\,\gamma\in\Gamma$, $a_\gamma=\perp$. In this case,

$$\langle a_\gamma\rangle_{\gamma\in\Gamma}\; Q\; \langle b_\gamma\rangle_{\gamma\in\Gamma} \;\Leftrightarrow\; \exists\,\beta\in\Gamma,\, b_\beta=\perp.$$

Take $\coprod_{\gamma\in\Gamma} A_\gamma\,/\,Q$ to be the corresponding quotient; and put $\preceq$ on $\coprod_{\gamma\in\Gamma} A_\gamma\,/\,Q$ by saying that $\left[\langle\perp_\gamma\rangle_{\gamma\in\Gamma}\right]\preceq$ all classes, and otherwise $\preceq$ is the trivial extension of the original order $\le$. Let $q:\coprod_{\gamma\in\Gamma} A_\gamma\to\coprod_{\gamma\in\Gamma} A_\gamma\,/\,Q$ be the quotient map.

3.5.1 Theorem. The following hold:

1. Q is an equivalence relation.
2. $\widehat{\coprod}_{\gamma\in\Gamma} A_\gamma\equiv\coprod_{\gamma\in\Gamma} A_\gamma\,/\,Q$, with the order $\preceq$, is an object in $\mathbf{SLat}_\perp(\wedge)$, i.e., $\widehat{\coprod}_{\gamma\in\Gamma}A_\gamma$ is a meet semilattice with bottom $\left[\langle\perp_\gamma\rangle_{\gamma\in\Gamma}\right]$.
3. The quotient map $q:\coprod_{\gamma\in\Gamma} A_\gamma\to\widehat{\coprod}_{\gamma\in\Gamma} A_\gamma$ is a $\mathbf{SLat}(\wedge)$ epimorphism and each $q\circ\varphi_\beta:A_\beta\to\widehat{\coprod}_{\gamma\in\Gamma} A_\gamma$ is a $\mathbf{SLat}_\perp(\wedge)$ morphism.
4. The quotient map q is universal with respect to all $\mathbf{SLat}(\wedge)$ morphisms from $\coprod_{\gamma\in\Gamma} A_\gamma$ to $\mathbf{SLat}_\perp(\wedge)$ objects.
5. $\left(\widehat{\coprod}_{\gamma\in\Gamma} A_\gamma, \{q\circ\varphi_\gamma\}_{\gamma\in\Gamma}\right)$ is the coproduct of $\{A_\gamma\}_{\gamma\in\Gamma}$ in $\mathbf{SLat}_\perp(\wedge)$.

We now have a precise counterpart $\left(\widehat{\coprod}_{\gamma\in\Gamma} A_\gamma, \{q\circ\varphi_\gamma\}_{\gamma\in\Gamma}\right)$ to the basis of the traditional product topology together with the usual injections of the factor topologies into that basis.

3.6 Step 3: First attempt to "close up" the "basis" $\widehat{\coprod}_{\gamma\in\Gamma} A_\gamma$. Apply the Downset Functor Theorem in Tool 1 to Step 2 to create the free frame $\mathcal{D}\left(\widehat{\coprod}_{\gamma\in\Gamma} A_\gamma\right)$ along with the unit

$$\eta_{\widehat{\coprod}_{\gamma\in\Gamma} A_\gamma}:\widehat{\coprod_{\gamma\in\Gamma}} A_\gamma\to\mathcal{D}\left(\widehat{\coprod_{\gamma\in\Gamma}} A_\gamma\right),$$

and consider the compositions

$$\eta_{\widehat{\coprod}_{\gamma\in\Gamma} A_\gamma}\circ q\circ\varphi_\beta:A_\beta\to\mathcal{D}\left(\widehat{\coprod_{\gamma\in\Gamma}} A_\gamma\right).$$

These compositions preserve finite meets and bottom (the empty join) and possibly *some* nonempty joins; so that $\left(\mathcal{D}\left(\widehat{\coprod}_{\gamma\in\Gamma} A_\gamma\right), \left\{\eta_{\widehat{\coprod}_{\gamma\in\Gamma} A_\gamma}\circ q\circ\varphi_\gamma\right\}_{\gamma\in\Gamma}\right)$ acts *somewhat* like a product topology, i.e., *somewhat* like the **Frm** coproduct of $\{A_\gamma\}_{\gamma\in\Gamma}$.

3.6.1 Question. Let $M\neq\varnothing$ be any nonempty set, fix $\beta\in\Gamma$, and let $\{a_m:m\in M\}\subset A_\beta$. Is it necessarily the case that

$$\left(\eta_{\widehat{\coprod}_{\gamma\in\Gamma} A_\gamma}\circ q\circ\varphi_\beta\right)\left(\bigvee_{m\in M} a_m\right)=\bigvee_{m\in M}\left(\eta_{\widehat{\coprod}_{\gamma\in\Gamma} A_\gamma}\circ q\circ\varphi_\beta\right)(a_m)\,?$$

Whenever the answer is "yes", it means that this "union of basic open subsets" has been "closed up". But the answer need not be always "yes", leading us to Step 4 below.

3.7 Step 4: Second attempt to "close up" the "basis". We apply the Frame Quotient Theorem in Tool 2 to Step 3. Put on $\mathcal{D}\left(\widehat{\coprod}_{\gamma\in\Gamma} A_\gamma\right)$ the binary relation R that essentially relates the two sides of Question 3.6.1; namely, let R be given by

$$\left\{ \begin{array}{c} \left(\left(\eta_{\widehat{\coprod}_{\gamma\in\Gamma} A_\gamma} \circ q \circ \varphi_\beta\right)\left(\bigvee_{m\in M} a_m\right), \bigvee_{m\in M}\left(\eta_{\widehat{\coprod}_{\gamma\in\Gamma} A_\gamma} \circ q \circ \varphi_\beta\right)(a_m)\right) : \\ \beta\in\Gamma,\ M\neq\varnothing,\ \{a_m : m\in M\}\subset A_\beta \end{array} \right\}.$$

Now the Frame Quotient Theorem creates the quotient frame

$$\mathcal{D}\left(\widehat{\coprod}_{\gamma\in\Gamma} A_\gamma\right) / R$$

along with the nucleus $\nu_R : \mathcal{D}\left(\widehat{\coprod}_{\gamma\in\Gamma} A_\gamma\right) \to \mathcal{D}\left(\widehat{\coprod}_{\gamma\in\Gamma} A_\gamma\right) / R$. Put

$$\bigoplus_{\gamma\in\Gamma} A_\gamma \equiv \mathcal{D}\left(\widehat{\coprod}_{\gamma\in\Gamma} A_\gamma\right) / R,$$

$$\iota_\beta : A_\beta \to \bigoplus_{\gamma\in\Gamma} A_\gamma \quad \text{by} \quad \iota_\beta \equiv \nu_R \circ \eta_{\widehat{\coprod}_{\gamma\in\Gamma} A_\gamma} \circ q \circ \varphi_\beta.$$

3.7.1 Theorem. The following hold:

1. Each ι_β preserves arbitrary $\bigvee$ and finite $\wedge$, i.e., is a frame morphism.
2. $\left(\bigoplus_{\gamma\in\Gamma} A_\gamma,\ \{\iota_\gamma\}_{\gamma\in\Gamma}\right)$ is the coproduct in **Frm** of $\{A_\gamma\}_{\gamma\in\Gamma}$, i.e., $\left(\bigoplus_{\gamma\in\Gamma} A_\gamma,\ \{\iota_\gamma^{op}\}_{\gamma\in\Gamma}\right)$ is the product in **Loc** of $\{A_\gamma\}_{\gamma\in\Gamma}$.

Proof. Clearly $\bigoplus_{\gamma\in\Gamma} A_\gamma$ is a frame. For each $\beta\in\Gamma$, each of $\nu_R, \eta_{\widehat{\coprod}_{\gamma\in\Gamma} A_\gamma}, q, \varphi_\beta$ preserves finite meets, so that ι_β does; each of $\nu_R, \eta_{\widehat{\coprod}_{\gamma\in\Gamma} A_\gamma}, q\circ\varphi_\beta$ preserves bottom (even though φ_β does not if $|\Gamma|\geq 2$—see 3.8 below), so that ι_β; by choice of the relationship R on $\mathcal{D}\left(\widehat{\coprod}_{\gamma\in\Gamma} A_\gamma\right)$, ι_β preserves arbitrary nonempty joins; and so $\{\iota_\gamma\}_{\gamma\in\Gamma}$ is a family of frame maps from the A_γ's to $\bigoplus_{\gamma\in\Gamma} A_\gamma$.

Now let $\left(B,\ \{i_\gamma\}_{\gamma\in\Gamma}\right)$ be given with B a frame and $\{i_\gamma\}_{\gamma\in\Gamma}$ a collection of frame maps from from the A_γ's to B. The couniversality of the φ_γ's forces the existence of a meet semilattice morphism

$$h_1 : \coprod_{\gamma\in\Gamma} A_\gamma \to B$$

which factors the i_γ's through the φ_γ's; the couniversality of q then forces the existence of a bottom preserving meet semilattice morphism

$$h_2 : \widehat{\coprod}_{\gamma\in\Gamma} A_\gamma \to B$$

which factors h_1 through q; the universality of $\eta_{\widehat{\coprod}_{\gamma\in\Gamma} A_\gamma}$ with respect to bottom preserving meet semilattice morphisms then forces the existence of a frame map

$$h_3 : \mathcal{D}\left(\widehat{\coprod}_{\gamma\in\Gamma} A_\gamma\right) \to B$$

which factors h_2 through $\eta_{\widehat{\coprod}_{\gamma\in\Gamma} A_\gamma}$; and, finally, the couniversality of ν_R then forces the existence of a frame map

$$h : \bigoplus_{\gamma\in\Gamma} A_\gamma \to B$$

which factors h_3 through ν_R. It follows

$$\begin{aligned} i_\beta &= h_1\circ\varphi_\beta = h_2\circ q\circ\varphi_\beta = h_3\circ\eta_{\widehat{\coprod}_{\gamma\in\Gamma} A_\gamma}\circ q\circ\varphi_\beta \\ &= h\circ\nu_R\circ\eta_{\widehat{\coprod}_{\gamma\in\Gamma} A_\gamma}\circ q\circ\varphi_\beta = h\circ\iota_\gamma, \end{aligned}$$

so that h factors the i_γ's through the ι_γ's.

As for the uniqueness of h, let $\widehat{h} : \bigoplus_{\gamma\in\Gamma} A_\gamma \to B$ be another frame map factoring the i_γ's through the ι_γ's. Then $\forall \beta \in \Gamma$, it is the case that

$$h \circ \nu_R \circ \eta_{\widehat{\coprod}_{\gamma\in\Gamma} A_\gamma} \circ q \circ \varphi_\beta = i_\beta = \widehat{h} \circ \nu_R \circ \eta_{\widehat{\coprod}_{\gamma\in\Gamma} A_\gamma} \circ q \circ \varphi_\beta.$$

Since $\{\varphi_\gamma\}_{\gamma\in\Gamma}$ is an epi-sink (3.4.2), then

$$h \circ \nu_R \circ \eta_{\widehat{\coprod}_{\gamma\in\Gamma} A_\gamma} \circ q = \widehat{h} \circ \nu_R \circ \eta_{\widehat{\coprod}_{\gamma\in\Gamma} A_\gamma} \circ q,$$

and the surjectivity of q (3.5.1(3)) yields

$$h \circ \nu_R \circ \eta_{\widehat{\coprod}_{\gamma\in\Gamma} A_\gamma} = \widehat{h} \circ \nu_R \circ \eta_{\widehat{\coprod}_{\gamma\in\Gamma} A_\gamma}.$$

Now the surjectivity of $\eta_{\widehat{\coprod}_{\gamma\in\Gamma} A_\gamma}$ with respect to a generating subset of $\mathcal{D}\left(\widehat{\coprod}_{\gamma\in\Gamma} A_\gamma\right)$ (3.2.1(5)) implies

$$h \circ \nu_R = \widehat{h} \circ \nu_R,$$

and the surjectivity of ν_R (3.3.1(1)) now gives

$$h = \widehat{h}. \quad \square$$

We may simply speak of $\bigoplus_{\gamma\in\Gamma} A_\gamma$ as the localic product of the A_γ's.

3.8 Discussion. The insertion of Step 2 into the construction of the localic product is mandatory and deserves some discussion. There are two main points, the first dealing with a possible family of frame maps as a candidate for injections, and the second dealing with whether this candidate is couniversal—and it is the second point which is unavoidably critical.

1. Step 2 interposes q to clean up the behavior of the $\{\varphi_\gamma\}_{\gamma\in\Gamma}$ with respect to preservation of bottom elements: this yields $\left(\widehat{\coprod}_{\gamma\in\Gamma} A_\gamma, \{q \circ \varphi_\gamma\}_{\gamma\in\Gamma}\right)$ as a precise counterpart to the basis of a product topology as well as allows for a smaller relation R in Step 4 defined using nonempty sets M. On the other hand, if one were to bypass Step 2 and apply the Downset Functor Theorem directly to Step 1 to get Step 3 (without Step 2) and then apply the Frame Quotient Theorem to Step 3 to get Step 4, then the members of the family $\left\{\nu_R \circ \eta_{\coprod_{\gamma\in\Gamma} A_\gamma} \circ \varphi_\gamma\right\}_{\gamma\in\Gamma}$ fail to preserve bottom and fail to be frame maps and injections for the frame coproduct. This problem of preserving bottom can be rectified by enlarging the relation R in Step 4 to allow the indexing set M to be nonempty; but the "aesthetic" problem of $\coprod_{\gamma\in\Gamma} A_\gamma$ not matching the basis of a product topology remains as well as the "real" problem of couniversality of the family $\left\{\nu_R \circ \eta_{\coprod_{\gamma\in\Gamma} A_\gamma} \circ \varphi_\gamma\right\}_{\gamma\in\Gamma}$, the latter being addressed in our next point. To conclude this point, it is both more efficient and topologically intuitive to insert Step 2 into the construction.

2. Continuing the discussion from (1), let us try to prove that $\left\{\nu_R \circ \eta_{\coprod_{\gamma\in\Gamma} A_\gamma} \circ \varphi_\gamma\right\}_{\gamma\in\Gamma}$ is couniveral with respect to the frame constructed in Steps 1, 3, 4 *sans* Step 2, call it A, with the assumption that the relation R in Step 4 allows for empty M so that $\left\{\nu_R \circ \eta_{\coprod_{\gamma\in\Gamma} A_\gamma} \circ \varphi_\gamma\right\}_{\gamma\in\Gamma}$ is a collection of frame maps. Suppose that $\{i_\gamma\}_{\gamma\in\Gamma}$ is another family of maps from the A_γ's to a frame B. For $\left\{\nu_R \circ \eta_{\coprod_{\gamma\in\Gamma} A_\gamma} \circ \varphi_\gamma\right\}_{\gamma\in\Gamma}$ to be couniversal, we must show there is a unique map $h : A \to B$ such that $\forall \beta \in \Gamma$, we have
$$i_\beta = h \circ \nu_R \circ \eta_{\coprod_{\gamma\in\Gamma} A_\gamma} \circ \varphi_\beta.$$
As in the proof of 3.7.1, we have that B is a meet-semilattice, so that the couniversality of the $\{\varphi_\gamma\}_{\gamma\in\Gamma}$ forces the existence of a unique meet semilattice morphism
$$h_1 : \coprod_{\gamma\in\Gamma} A_\gamma \to B$$

such that $\forall \gamma \in \Gamma$, we have

$$i_\gamma = h_1 \circ \varphi_\gamma.$$

Now we want to continue this process to say that $h_1 : \coprod_{\gamma \in \Gamma} A_\gamma \to B$ factors uniquely through

$$\eta_{\coprod_{\gamma \in \Gamma} A_\gamma} : \coprod_{\gamma \in \Gamma} A_\gamma \to \mathcal{D}\left(\coprod_{\gamma \in \Gamma} A_\gamma\right)$$

via some frame map

$$h_2 : \mathcal{D}\left(\coprod_{\gamma \in \Gamma} A_\gamma\right) \to B.$$

But since we only know that $\eta_{\coprod_{\gamma \in \Gamma} A_\gamma}$ is universal with respect to meet semilattice maps that preserve *bottom*, we know we have the factoring map h_2 if and only if we know that h_1 preserves *bottom*. But we can know that h_1 preserves bottom if and only if $|\Gamma| = 1$. Sufficiency is trivial. To see necessity, we note that $i_\gamma = h_1 \circ \varphi_\gamma$ is a frame map and preserves bottom. But if $|\Gamma| \geq 2$, then each φ_γ does not preserve bottom and in fact cannot produce bottom as an output: let $\gamma_1 \neq \gamma_2$ in Γ, $a \neq \perp \in A_{\gamma_1}$, and $\top \in A_{\gamma_2}$; then

$$\varphi_{\gamma_2}(\perp) = \begin{cases} \perp, & \gamma = \gamma_1 \\ \top_\gamma, & \gamma \neq \gamma_1 \end{cases} \neq \langle \perp_\gamma \rangle_{\gamma \in \Gamma},$$

$$\varphi_{\gamma_2}(a) = \begin{cases} a, & \gamma = \gamma_1 \\ \top_\gamma, & \gamma \neq \gamma_1 \end{cases} \neq \langle \perp_\gamma \rangle_{\gamma \in \Gamma}.$$

Therefore we can never test h_1 for preservation of bottom using $i_\gamma = h_1 \circ \varphi_\gamma$, and therefore we can never know whether h_1 preserves bottom. Therefore we cannot know whether the universality of $\eta_{\coprod_{\gamma \in \Gamma} A_\gamma}$ can be applied to get h_2, and therefore the proof of the couniversality of the family $\left\{\nu_R \circ \eta_{\coprod_{\gamma \in \Gamma} A_\gamma} \circ \varphi_\gamma\right\}_{\gamma \in \Gamma}$ cannot be continued, and this means that we do not have the localic product (without Step 2) regardless of how R is chosen in Step 4. The only solution to this impasse is to insert Step 2 between Step 1 and Step 3 to then insure that $\eta_{\widehat{\coprod}_{\gamma \in \Gamma} A_\gamma}$ is only used to factor meet semilattice morphisms which preserve bottom. Therefore, it is mandatory to insert Step 2 into the localic product construction of [13].

3.9 Question. Now that we have the localic product clearly and fully in hand, we may ask, in the case when the locales are traditional or L-topologies, how the localic product compares order-theoretically with the traditional or L-valued topological product, respectively? This is Question 1.1 and the subject of subsequent sections of this paper.

In connection with 3.9, we have the following class of examples from [9]:

3.9.1 Example Class. Let X, Y be dense subspaces of a regular topological space such that $X \cap Y$ is not dense in this space, and let $\mathfrak{T}_X, \mathfrak{T}_Y$ respectively denote the subspace topologies. Then $\mathfrak{T}_X \oplus \mathfrak{T}_Y$ is not spatial and hence by Theorem 1.2, $\mathfrak{T}_X \oplus \mathfrak{T}_Y \ncong \mathfrak{T}_X \otimes \mathfrak{T}_Y$. As a special instance of this example class, $\mathfrak{T}_\mathbb{Q} \oplus \mathfrak{T}_\mathbb{Q}$ is not spatial and $\mathfrak{T}_\mathbb{Q} \oplus \mathfrak{T}_\mathbb{Q} \ncong \mathfrak{T}_\mathbb{Q} \otimes \mathfrak{T}_\mathbb{Q}$.

3.10 Remark. It should be pointed that not only does the localic product help secure the completeness of **Loc** in general, but it is also useful for the explicit construction of many important structures and functors.

1. Pullbacks in **Loc**, or pushouts in **Frm**, are explicitly constructed using localic products together with the Frame Quotient Theorem 3.3.1. This is an instance of the more general phenomenon in category theory of pushouts being constructed using coproducts and coequalizers.

2. The construction of the important left-adjoint of the upper forgetful functor from L-**Frm** to **Frm**, for L a complete chain, makes unavoidable use of the localic product—see [16, 17].

4 $L\Omega \dashv LPT$ and proofs of Theorems 1.4, 1.5, 1.6

That $\{L\text{-}\mathbf{Top} : L \in |\mathbf{SFrm}|\}$ furnishes representations for all (semi)locales is well-known from [3, 5, 6, 10, 11, 14, 15, 16, 17, 18, 19, 20, 22, 24, 26], from which the following paragraph and all unreferenced and unproven statements of this section are summarzed.

Fix $L \in |\mathbf{Frm}|$. Let $A \in |\mathbf{Loc}|$ and put

$$\begin{aligned} Lpt\,(A) &= \mathbf{Frm}\,(A, L) \\ &= \left\{p : A \to L \mid p \text{ preserves arbitrary} \bigvee \text{and finite} \wedge \right\} \end{aligned}$$

and define the "first comparison map"

$$\Phi_L : A \to L^{Lpt(A)} \quad \text{by} \quad \Phi_L\,(a)\,(p) = p\,(a)\,.$$

It is well-known that Φ_L is a frame morphism; hence

$$LPT\,(A) \equiv (Lpt\,(A)\,, (\Phi_L)^{\rightarrow}\,(A))$$

is an L-topological space called the *L-spectrum* of A. Since τ is a locale for each L-topological space (X, τ), we may form the L-spectrum $(Lpt\,(\tau)\,, (\Phi_L)^{\rightarrow}\,(\tau))$ of τ and compare it against the original space (X, τ) via the "second comparison map" $\Psi_L : X \to Lpt\,(\tau)$ defined by

$$\Psi_L(x) : \tau \to L \quad \text{via} \quad \Psi_L(x)(u) = u(x).$$

Given $f : A \to B$ in **Loc**, we set

$$LPT\,(f) : LPT\,(A) \to LPT\,(B) \quad \text{by} \quad LPT\,(f)\,(p) = p \circ f^{op}.$$

Altogether we have a functor $LPT : \mathbf{Loc} \to L\text{-}\mathbf{Top}$. Now in the opposite direction we have the functor $L\Omega : L\text{-}\mathbf{Top} \to \mathbf{Loc}$ given by

$$L\Omega\,(X, \tau) = \tau, \quad L\Omega\,[f : (X, \tau) \to (Y, \sigma)] = \left[(f_L^{\leftarrow})_{|\,\sigma}\right]^{op} : \tau \to \sigma.$$

4.1 Definition. A locale A is *L-spatial* if Φ_L is injective; an L-topological space (X, τ) is *L-sober* if Ψ_L is bijective; (X, τ) is L-T_0 if Ψ_L is injective; (X, τ) is L-S_0 if Ψ_L is surjective; and (X, τ) is *quasi*-L-S_0 (q-L-S_0) if

$$\{p \in Lpt\,(\tau) : coker\,(p) \neq \tau - \{\bot\}\} \subset (\Psi_L)^{\rightarrow}\,(X)\,,$$

where $coker(p) \equiv \{u \in \tau : p\,(u) = \top\}$. The L may be dropped if understood.

The L-S_0 axiom ignores the issue of being L-T_0. There can be at most one $p \in Lpt\,(\tau)$ with $coker\,(p) = \tau - \{\bot\}$; so (X, τ) being q-L-S_0 says Ψ_L is surjective *except* for at most one (L-)point on τ; and if there should be such a point, it would be **2**-valued and the maximal point on τ with the pointwise ordering. For crisp topological spaces (with $L = \mathbf{2}$), the q-L-S_0 condition says each irreducible closed set, *except* the carrier set (whole space), is the closure of a singleton. Clearly, L-sober implies L-T_0 and L-S_0, and L-S_0 implies q-L-S_0.

1. Many L-T_0 and q-L-S_0 spaces are not L-S_0 (and not L-sober). For $L = \mathbf{2}$, each infinite set X with the cofinite topology $\mathfrak{T}_{cof}$ is T_0 (in fact, T_1) and not S_0—X is irreducible closed, but not the closure of a (unique) singleton. Further, $(X, \mathfrak{T}_{cof})$ is q-L-S_0 since every irreducible closed subset other than X is the closure of a singleton.

2. Many L-S_0 spaces are not L-T_0 (and not L-sober). For $L = \mathbf{2}$, each indiscrete space $(Y, \mathfrak{I})$ with at least two elements is L-S_0 and not L-T_0 (and not L-sober).

3. Many q-L-S_0 spaces are not L-T_0. For $L = \mathbf{2}$, and using $(X, \mathfrak{T}_{cof})$ from (1) and $(Y, \mathfrak{I})$ from (2), the topological product $(X \times Y, \mathfrak{T}_{cof} \otimes \mathfrak{I})$ is q-L-S_0 and not L-T_0 (and not L-S_0).

4. The examples of (1), (2), (3) are carried in L-**Top** for $L \neq \mathbf{2}$ by the characteristic functor G_χ. The reader may construct other examples in a given L-**Top** not generated by G_χ.

4.2 Theorem. The following hold:

1. $\Psi_L : (X, \tau) \to LPT(\tau)$ is continuous and relatively open.
2. $L\Omega \dashv LPT$ with unit Ψ_L and counit $\Phi_L^{op} : A \leftarrow (\Phi_L)^{\to}(A)$ (in **Loc**).
3. Each $L\Omega(X, \tau) = \tau$ is L-spatial.
4. A locale A is L-spatial $\Leftrightarrow$ $\Phi_L : A \to (\Phi_L)^{\to}(A)$ is a frame isomorphism $\Leftrightarrow$ there is $(X, \tau) \in |L\text{-}\mathbf{Top}|$ with $\tau \cong A$.
5. An L-topological space (X, τ) is L-T_0 $\Leftrightarrow$ $\forall x, y \in X$ with $x \neq y$, $\exists u \in \tau$, $u(x) \neq u(y)$ $\Leftrightarrow$ Ψ_L is an L-embedding.
6. An L-topological space (X, τ) is L-S_0 $\Leftrightarrow$ Ψ_L is continuous and open.
7. An L-topological space (X, τ) is L-sober $\Leftrightarrow$ Ψ_L is an L-homeomorphism.
8. Each L-spectrum $LPT(A)$ is L-sober.
9. $L\Omega \sim LPT$ and L-**SobTop** $\sim$ L-**SpatLoc**, when $L\Omega \dashv LPT$ is restricted, respectively, to these categories: the full subcategory of L-**Top** of all L-sober (L-)topological spaces and the full subcategory of **Loc** of all L-spatial locales.

Proof. All these statements and their proofs are well-known (e.g., [20, 22]) or straightforward, except possibly the second equivalence of (4). For necessity of the second equivalence of (4), note that $(Lpt(A), (\Phi_L)^{\to}(A))$ is an L-topological space and apply (3) together with the first equivalence of (4). Now for sufficiency of the second equivalence of (4), note LPT is a functor and so preserves isomorphisms; hence, $LPT(A)$ is L-homeomorphic to $LPT(\tau)$. It now follows from 2.4.2(3) that $(\Phi_L)^{\to}(A) \cong (\Phi_L)^{\to}(\tau)$; also $(\Phi_L)^{\to}(\tau) \cong \tau$ by (3) and the first equivalence of (4), and $\tau \cong A$ by assumption. Therefore, $(\Phi_L)^{\to}(A) \cong A$. $\square$

The situation when M is a subframe of L, including the cases $M = L$ or $M = \mathbf{2}$ (which requires L to be consistent), yields the following extension—essentially corollary—of 4.2 needed in Sections 5 and 6.

4.2.1 Theorem. Let M be a subframe of L. The following hold:

1. $\forall M$ subframe of L, $[\forall (X, \tau) \in |L\text{-}\mathbf{Top}|, L\Omega(X, \tau) = \tau$ is M-spatial$]$ $\Leftrightarrow$ L is M-spatial. In particular, L is L-spatial.
2. L is M-spatial $\Leftrightarrow$ M-**SpatLoc** $=$ L-**SpatLoc** $\Leftrightarrow$ M-**SobTop** $\sim$ L-**SobTop**.
3. If L is M-spatial, then M-**Top** is a full subcategory of L-**Top** and the L-topological product (in L-**Top**) of M-topological spaces is precisely the M-topological product (in M-**Top**) of these spaces.
4. If L is M-spatial, then MPT preserves all localic products to L-topological products (in L-**Top**) of M-topological spaces.

Proof. *Ad (1).* Let M be a subframe of L. For necessity, let $X = \mathbf{1}$ (singleton) and set $\tau = L^{\mathbf{1}}$. Then by assumption, τ is M-spatial. But $L \cong L^{\mathbf{1}}$. Using the last equivalence of 4.2(4), L is M-spatial. For sufficiency, suppose L is M-spatial, and let $u \neq v$ in τ. By 4.2(3), τ is L-spatial, so that $\Phi_L : \tau \to (\Phi_L)^{\to}(\tau)$ is injective and so $\Phi_L(u) \neq \Phi_L(v)$. Since $(\Phi_L)^{\to}(\tau) \subset L^{Lpt(\tau)}$, this means there is frame map $p : \tau \to L$, $p(\Phi_L(u)) \neq p(\Phi_L(v))$. But L is M-spatial, so by similar argumentation, there is a frame map $q : L \to M$, $q(p(\Phi_L(u))) \neq q(p(\Phi_L(v)))$. Since $q \circ p : \tau \to M$ is a frame map, reversing this argumentation yields that $\Phi_M : \tau \to (\Phi_M)^{\to}(\tau)$ is injective, and so τ is M-spatial. The proof of the first statement of (1) yields the second statement of (1) using the L-topological space $\left(\mathbf{1}, L^{\mathbf{1}}\right)$.

Ad (2). Suppose L is M-spatial. Then by (1) we have M-**SpatLoc** $=$ L-**SpatLoc**; and it then follows that

$$M\text{-}\mathbf{SobTop} \sim M\text{-}\mathbf{SpatLoc} = L\text{-}\mathbf{SpatLoc} \sim L\text{-}\mathbf{SobTop}.$$

On the other hand if we assume M-**SobTop** $\sim$ L-**SobTop**, then we must have M-**SpatLoc** $\sim$ L-**SpatLoc**. This means that there is an adjunction between these categories whose units and counits are isomorphisms in their respective categories. In particular, each L-spatial locale—and hence every L-topology—is order-isomorphic to some M-spatial locale and is therefore M-spatial (4.2(3,4)). Applying (1) yields that L is M-spatial.

Ad (3). Since M is a subframe of L, each M-topological space is clearly an L-topological space. Further, we observe that

$$f_M^{\leftarrow} = \left(f_L^{\leftarrow}\right)_{|\,M^X},$$

from which it follows that a ground map between the carrier sets of two M-topological spaces is M-continuous if and only if it is L-continuous. The claims of (3) now follow.

Ad (4). Because of 4.2(3,4), it suffices to let $\{(X_\gamma, \tau_\gamma) : \gamma \in \Gamma\}$ be a family of L-topological spaces. Since L is M-spatial, then each τ_γ is M-spatial by (1). Thus by 4.2(4), $\forall \gamma \in \Gamma$, $\exists\ (Y_\gamma, \sigma_\gamma) \in M$-**Top** such that $\sigma_\gamma \cong \tau_\gamma$. Since $\bigoplus$ is the categorical product for **Loc**, it follows that

$$\bigoplus_{\gamma\in\Gamma} \sigma_\gamma \cong \bigoplus_{\gamma\in\Gamma} \tau_\gamma.$$

And since MPT is a functor, the following hold: $\forall \gamma \in \Gamma$,

$$\begin{aligned}\left(Mpt\left(\sigma_\gamma\right), \left(\Phi_M\right)^{\rightarrow}\left(\sigma_\gamma\right)\right) &= MPT\left(\sigma_\gamma\right) \\ &\cong MPT\left(\tau_\gamma\right) \\ &= \left(Mpt\left(\tau_\gamma\right), \left(\Phi_M\right)^{\rightarrow}\left(\tau_\gamma\right)\right),\end{aligned}$$

and also

$$MPT\left(\bigoplus_{\gamma\in\Gamma}\sigma_\gamma\right) \cong MPT\left(\bigoplus_{\gamma\in\Gamma}\tau_\gamma\right),$$

all "$\cong$"in the sense of L-homeomorphisms (by (3)). Now the adjunction $M\Omega \dashv MPT$ assures us that MPT preserves localic products to M-topological products, in which case we have

$$\begin{aligned}MPT\left(\bigoplus_{\gamma\in\Gamma}\tau_\gamma\right) &\cong MPT\left(\bigoplus_{\gamma\in\Gamma}\sigma_\gamma\right) \\ &= \left(\prod_{\gamma\in\Gamma} Mpt\left(\sigma_\gamma\right), \bigotimes_{\gamma\in\Gamma}\left(\Phi_M\right)^{\rightarrow}\left(\sigma_\gamma\right)\right) \\ &\cong \left(\prod_{\gamma\in\Gamma} Mpt\left(\tau_\gamma\right), \bigotimes_{\gamma\in\Gamma}\left(\Phi_M\right)^{\rightarrow}\left(\tau_\gamma\right)\right).\end{aligned}$$

But all these products may be taken as L-topological products, concluding the proof of (4) and the theorem. $\square$

It should be noted that the classical representation of spatial locales by sober topological spaces is obtained from the foregoing, via the characteristic functor G_χ, by restricting $L = \mathbf{2}$. More specifically, referring to the Ω : **Top** $\rightarrow$ **Loc** and PT : **Top** $\leftarrow$ **Loc** functors of [9] and the associated terminology of Pt, Φ, Ψ, we have—using and mixing G_χ (and its inverse) at both the fibre and categorical level—

$$Pt = \mathbf{2}pt, \quad \Phi = G_\chi \circ \Phi_{\mathbf{2}}, \quad \Psi = G_\chi^{-1} \circ \Psi_{\mathbf{2}} \circ G_\chi,$$

$$\Omega = G_\chi^{-1} \circ L\Omega \circ G_\chi, \quad PT = \Omega = G_\chi \circ \mathbf{2}PT,$$

$$L\Omega \dashv LPT \text{ with unit } \Psi \text{ and counit } \Phi_L^{op}.$$

Further, noting that spatiality is **2**-spatiality, and sobriety of $(X, \mathfrak{T})$ is logically equivalent to **2**-sobriety of $G_\chi(X, \mathfrak{T})$, we now have **SobTop** $\sim$ **SpatLoc** via the restriction of $\Omega \dashv PT$ to these subcategories.

4.2.2 Terminology.

1. $LPT \circ L\Omega : L$-**Top** $\rightarrow L$-**SobTop** is the *L-soberification functor*, $LPT \circ M\Omega : M$-**Top** $\rightarrow L$-**SobTop** is the *L-M soberification functor*, and $MPT \circ L\Omega : L$-**Top** $\rightarrow M$-**SobTop** is the *M-L soberification functor*. These terms are justified by 4.2 and 4.2.1 above.
2. We refer only to L-continuity and L-homeomorphisms under the conditions of 4.2.1(3).

4.3 Proof of Theorem 1.4. Assume $\bigoplus_{\gamma\in\Gamma}\tau_\gamma \cong \bigotimes_{\gamma\in\Gamma}\tau_\gamma$. Now since $\bigotimes_{\gamma\in\Gamma}\tau_\gamma$ is an L-topology, the second equivalence of 4.2(4) now applies to say $\bigoplus_{\gamma\in\Gamma}\tau_\gamma$ is L-spatial. □

4.4 Proof of Theorem 1.5. The first statement of 1.5 follows from the second statement of 1.5 using 4.2(5) and choosing $M = L$. We now show the second statement of 1.5. Assume that M is a subframe of L and L is M-spatial; and assume $\bigoplus_{\gamma\in\Gamma}\tau_\gamma \cong \bigotimes_{\gamma\in\Gamma}(\Phi_M)^{\rightarrow}(\tau_\gamma)$. Then the M-spatiality of $\bigoplus_{\gamma\in\Gamma}\tau_\gamma$ follows from 4.2(4) since the L-topology $\bigotimes_{\gamma\in\Gamma}(\Phi_M)^{\rightarrow}(\tau_\gamma)$ is M-spatial by 4.2.1(1).

Now suppose for sufficiency that $\bigoplus_{\gamma\in\Gamma}\tau_\gamma$ is L-spatial. Then 4.2.1(2) implies that $\bigoplus_{\gamma\in\Gamma}\tau_\gamma$ is M-spatial. Now 4.2.1(4) states that MPT preserves products from **Loc** to L-**Top**; and this implies that

$$\left(Mpt\left(\bigoplus_{\gamma\in\Gamma}\tau_\gamma\right), (\Phi_M)^{\rightarrow}\left(\bigoplus_{\gamma\in\Gamma}\tau_\gamma\right)\right) \cong \left(\prod_{\gamma\,\in\,\Gamma} Mpt\,(\tau_\gamma)\,, \bigotimes_{\gamma\in\Gamma}(\Phi_M)^{\rightarrow}(\tau_\gamma)\right),$$

"$\cong$" refers to L-homeomorphism (4.2.1(3)), which by 2.4.2(3) implies that

$$(\Phi_M)^{\rightarrow}\left(\bigoplus_{\gamma\in\Gamma}\tau_\gamma\right) \cong \bigotimes_{\gamma\in\Gamma}(\Phi_M)^{\rightarrow}(\tau_\gamma)\,.$$

Now from the M-spatiality of $\bigoplus_{\gamma\in\Gamma}\tau_\gamma$, it follows from the first equivalence of 4.2(4) that

$$\bigoplus_{\gamma\in\Gamma}\tau_\gamma \cong (\Phi_M)^{\rightarrow}\left(\bigoplus_{\gamma\in\Gamma}\tau_\gamma\right),$$

forcing $\bigoplus_{\gamma\in\Gamma}\tau_\gamma \cong \bigotimes_{\gamma\in\Gamma}(\Phi_M)^{\rightarrow}(\tau_\gamma)\,.$ □

4.5 Proof of Theorem 1.6. Because of 4.3, we need only prove sufficiency. From 4.4 we already have that $\bigoplus_{\gamma\in\Gamma}\tau_\gamma \cong \bigotimes_{\gamma\in\Gamma}(\Phi_L)^{\rightarrow}(\tau_\gamma)$. Now since each (X_γ, τ_γ) is L-sober, it follows from 4.2(7) that

$$\forall\gamma\in\Gamma,\ (X_\gamma,\tau_\gamma) \cong \left(Lpt\,(\tau_\gamma)\,, (\Phi_L)^{\rightarrow}(\tau_\gamma)\right),$$

$$\left(\prod_{\gamma\,\in\,\Gamma} X_\gamma, \bigotimes_{\gamma\in\Gamma}\tau_\gamma\right) \cong \left(\prod_{\gamma\,\in\,\Gamma} Lpt\,(\tau_\gamma)\,, \bigotimes_{\gamma\in\Gamma}(\Phi_L)^{\rightarrow}(\tau_\gamma)\right).$$

And 2.4.2(3) now applies to say that

$$\bigotimes_{\gamma\in\Gamma}\tau_\gamma \cong \bigotimes_{\gamma\in\Gamma}(\Phi_L)^{\rightarrow}(\tau_\gamma)\,.$$

Hence $\bigoplus_{\gamma\in\Gamma}\tau_\gamma \cong \bigotimes_{\gamma\in\Gamma}\tau_\gamma$. □

Theorem 1.6.1 generalizes Theorem 1.6 from L-sober spaces to L-S_0 spaces, and not assuming the L-T_0 axiom forces a quite different proof needing Section 6 for its set-up—see 6.12 below. The L-S_0 condition in 1.6.1 is further weakened in 1.7.1 to q-L-S_0 in the presence of prime separation and normalization by means of yet another and different proof in Section 8 below.

5 Separation conditions in L-topological products

This section identifies "separation conditions" related to how "distinct" factors in a product space are from each other. As it turns out, factors in traditional topological products and localic products do remain separated from each other in certain senses; but this is generally not the case for L-topological products when $|L| \geq 3$—such products can be quite "messy" and factors may "spill" over into each other in various ways. Since one of the primary goals of this paper is to compare localic products of L-topologies with L-topological products of L-topologies, restrictions under which L-topological products are better behaved with respect to these separation conditions play an essential role in subsequent sections.

We begin by cataloguing separation conditions satisfied by traditional topological product spaces and localic products. In the remainder of this paper, "AC" refers to the Axiom of Choice.

5.1 Fact (separation conditions in traditional topological products). Let $\{(X_\gamma, \mathfrak{T}_\gamma) : \gamma \in \Gamma\} \subset |\mathbf{Top}|$, let $U_\gamma, V_\gamma \in \mathfrak{T}_\gamma$ for each $\gamma \in \Gamma$, and let $\Lambda \subset \Gamma$.

1. Suppose Λ is finite and $\prod_{\lambda\in\Lambda} U_\lambda \neq \varnothing$. Then the following hold:
   (a) $\prod_{\lambda\in\Lambda} U_\lambda \subset \prod_{\lambda\in\Lambda} V_\lambda \Leftrightarrow \forall\lambda \in \Lambda, U_\lambda \subset V_\lambda$ (*product separation*).
   (b) $\forall\beta \in \Lambda$,
   $$\pi_\beta^{\rightarrow}\left(\prod_{\lambda\in\Lambda} U_\lambda\right) = U_\beta \quad \text{and} \quad \pi^{\rightarrow}_{\prod_{\gamma\neq\beta} X_\gamma}\left(\prod_{\lambda\in\Lambda} U_\lambda\right) = \left(\prod_{\lambda\neq\beta} U_\lambda\right)$$
   (*(upper) projection separation*).
2. Suppose $\prod_{\lambda\in\Lambda} V_\gamma \neq \prod_{\lambda\in\Lambda} X_\lambda$. Then the following hold:
   (a) $\sum_{\lambda\in\Lambda} U_\lambda \subset \sum_{\lambda\in\Lambda} V_\lambda \Leftrightarrow \forall\lambda \in \Lambda, U_\lambda \subset V_\lambda$ (*sum separation*), where "$\sum$" indicates the "star" or "sum" operation, namely
   $$\sum_{\lambda\in\Lambda} U_\lambda = \bigcup_{\lambda\in\Lambda}\left[U_\lambda \times \prod_{\beta\neq\lambda} U_\beta\right],$$
   and similarly for $\sum_{\gamma\in\Gamma} V_\gamma$.
   (b) $\forall\beta \in \Lambda$,
   $$(\pi_\beta)_\rightarrow\left(\sum_{\lambda\in\Lambda} V_\lambda\right) = V_\beta \quad \text{and} \quad \left(\pi_{\prod_{\gamma\neq\beta} X_\gamma}\right)_\rightarrow\left(\sum_{\lambda\in\Lambda} V_\lambda\right) = \left(\sum_{\lambda\neq\beta} V_\lambda\right)$$
   (*lower projection separation*), where $(\pi_\beta)_\rightarrow$ is the lower image operator of π_β.
3. If Λ is finite and each $V_\gamma \neq X_\gamma$, then $\sum_{\gamma\in\Gamma} V_\gamma \neq \prod_{\gamma\in\Gamma} X_\gamma$ and
   $$\prod_{\lambda\in\Lambda} U_\lambda \times \prod_{\gamma\in\Gamma-\Lambda} X_\gamma \subset \sum_{\gamma\in\Gamma} V_\gamma \Leftrightarrow \exists\,\lambda \in \Lambda, U_\lambda \subset V_\lambda$$
   (*prime separation*).

Each forward direction of (1)(a), (2)(a), 3(a) requires AC when Λ is infinite or if Γ is infinite and the products indexed over Λ are viewed as subsets of the product of all the X_γ's. The assumption $\prod_{\lambda\in\Lambda} U_\lambda \neq \varnothing$ $\left[\prod_{\lambda\in\Lambda} V_\gamma \neq \prod_{\lambda\in\Lambda} X_\lambda\right]$ can be replaced for finite Λ by assuming each $U_\lambda \neq \varnothing$ $[V_\lambda \neq X_\lambda]$; and for infinite Λ this replacement can be made under AC. Extensions of these properties for L-topological spaces are discussed in the sequel.

5.2 Product separation in localic products. Assuming the material and notation of Section 3 above, we take the following from [13]. Technically, we are really looking at a kind of "sum separation" in frame coproducts and hence a "product separation" in localic products, but retaining the addition notation of the frame coproduct. Put $\mathbb{O} \equiv \left[\langle\perp_\gamma\rangle_{\gamma\in\Gamma}\right]$ in $\bigoplus_{\gamma\in\Gamma} A_\gamma$, the equivalence class in Step 2 of Section 3 of the tuple in the frame product in which each coordinate is bottom. It is necessary to get a sense of how one takes a product (or sum) of elements of the A_γs in the localic product (5.2.2) before stating the product separation condition in the localic product.

5.2.1 Proposition. $U \in \bigoplus_{\gamma\in\Gamma} A_\gamma$ is saturated $\Leftrightarrow$ each of the following statements hold:

1. $\mathbb{O} \subset U$.
2. $\forall M \neq \varnothing, \forall m \in M, \forall \left[\langle a_\gamma^m\rangle_{\gamma\in\Gamma}\right] \in U, \forall \langle a_\gamma\rangle_{\gamma\in\Gamma} \in \coprod_{\gamma\in\Gamma} A_\gamma, \forall\beta \in \Gamma$ such that
   $$a_\gamma = \begin{cases} a_\gamma^m, & \gamma \neq \beta, \forall m \in M \\ \bigvee_{m\in M} a_\gamma^m, & \gamma = \beta \end{cases},$$
   it is the case that $\left[\langle a_\gamma\rangle_{\gamma\in\Gamma}\right] \in U$.

5.2.2 Proposition. Let $\left[\langle a_\gamma\rangle_{\gamma\in\Gamma}\right] \in \widehat{\coprod}_{\gamma\in\Gamma} A_\gamma$ and put

$$\oplus_{\gamma\in\Gamma} a_\gamma \equiv \downarrow\!\left(\langle a_\gamma\rangle_{\gamma\in\Gamma}\right) \cup \mathbb{O}.$$

Then $\oplus_{\gamma\in\Gamma} a_\gamma$ is saturated.

5.2.3 Corollary (product separation condition in localic products). Let $a_\gamma \neq \bot_\gamma$ for each $\gamma \in \Gamma$. Then $\forall \langle b_\gamma\rangle_{\gamma\in\Gamma} \in \coprod_{\gamma\in\Gamma} A_\gamma$,

$$\oplus_{\gamma\in\Gamma} a_\gamma \leq \oplus_{\gamma\in\Gamma} b_\gamma \;\Leftrightarrow\; \forall\gamma \in \Gamma,\; a_\gamma \leq b_\gamma.$$

We now define various separation conditions for L-topological products, including a finitary version of 5.2.3 (since as in 5.1 we are only interested in the basic open subsets of the L-product topology).

5.3 Definition (separation conditions for L-product topology). Let $\{(X_\gamma, \tau_\gamma) : \gamma \in \Gamma\}$ be a collection of L-topological spaces.

1. $\{(X_\gamma, \tau_\gamma) : \gamma \in \Gamma\}$ is *product separated* if $\forall n \in \mathbb{N}, \forall \{\gamma_i\}_{i=1}^n \subset \Gamma, [\forall i = 1, ..., n,\, a_{\gamma_i}, b_{\gamma_i} \in \tau_{\gamma_i}]$ with $\boxtimes_{i=1}^n a_{\gamma_i} \neq \bot$ (in $\bigotimes_{\gamma\in\Gamma} \tau_\gamma$),

$$\boxtimes_{i=1}^n a_{\gamma_i} \leq \boxtimes_{i=1}^n b_{\gamma_i} \;\Leftrightarrow\; \exists\, i = 1, ..., n,\; a_{\gamma_i} \leq b_{\gamma_i}\,.$$

2. $\{(X_\gamma, \tau_\gamma) : \gamma \in \Gamma\}$ is *sum separated* if $\forall B \subset \Gamma$, $[\forall\beta \in B,\, a_\beta, b_\beta \in \tau_\beta]$ with $\boxplus_{\beta\in B} b_\beta \neq \top$ (in $\bigotimes_{\gamma\in\Gamma} \tau_\gamma$),

$$\boxplus_{\beta\in B} a_\beta \leq \boxplus_{\beta\in B} b_\beta \Leftrightarrow \forall\beta \in B,\; a_\beta \leq b_\beta\,.$$

3. $\{(X_\gamma, \tau_\gamma) : \gamma \in \Gamma\}$ is *prime separated* if $\forall n \in \mathbb{N}$, $\forall \{\gamma_i\}_{i=1}^n \subset \Gamma$, $[\forall i = 1, ..., n,\, a_{\gamma_i} \in \tau_{\gamma_i}]$, $\forall B \subset \Gamma$ with $\{\gamma_i\}_{i=1}^n \subset B$, $[\forall\beta \in B,\, b_\beta \in \tau_\beta - \{\top\}]$, then $\boxplus_{\beta\in B} b_\beta \neq \top$ (in $\bigotimes_{\gamma\in\Gamma} \tau_\gamma$) and

$$\boxtimes_{i=1}^n a_{\gamma_i} \leq \boxplus_{\gamma\in\Gamma} b_{\gamma_i} \;\Leftrightarrow\; \exists\, i = 1, ..., n,\; a_{\gamma_i} \leq b_{\gamma_i}\,.$$

4. $\{(X_\gamma, \tau_\gamma) : \gamma \in \Gamma\}$ is *weakly prime separated* if $\forall n \in \mathbb{N}, \forall \{\gamma_i\}_{i=1}^n \subset \Gamma$, $[\forall i = 1, ..., n,\, a_{\gamma_i} \in \tau_{\gamma_i}]$, $\forall B \subset \Gamma$ with $\{\gamma_i\}_{i=1}^n \subset B$, $[\forall\beta \in B,\, b_\beta \in \tau_\beta - \{\top\}]$, then $\boxplus_{\beta\in B} b_\beta \neq \top$ (in $\bigotimes_{\gamma\in\Gamma} \tau_\gamma$) and

$$\boxtimes_{i=1}^n a_{\gamma_i} \leq \boxplus_{\gamma\in\Gamma} b_\gamma \;\Rightarrow\; \exists\, j = 1, ..., n,\; \boxtimes_{i=1}^n a_{\gamma_i} \leq b_{\gamma_j} \boxtimes \left(\boxtimes_{\gamma\neq\gamma_j} \top\right)\,,$$

   ($\top$ in $\bigotimes_{\gamma\neq\gamma_i} \tau_\gamma$).

5. $\{(X_\gamma, \tau_\gamma) : \gamma \in \Gamma\}$ is *upper projection separated* if $\forall n \in \mathbb{N}$, $\forall \{\gamma_i\}_{i=1}^n \subset \Gamma$, $[\forall i = 1, ..., n,\, a_{\gamma_i} \in \tau_{\gamma_i}]$ with $\boxtimes_{i=1}^n a_{\gamma_i} \neq \bot$ (in $\bigotimes_{\gamma\in\Gamma} \tau_\gamma$), $\forall j = 1, ..., n$,

$$\left(\pi_{\gamma_j}\right)_L^{\rightarrow} \left(\boxtimes_{i=1}^n a_{\gamma_i}\right) = a_{\gamma_j} \quad \text{and} \quad \left(\pi_{\prod_{\gamma\neq\gamma_j} X_\gamma}\right)_L^{\rightarrow} \left(\boxtimes_{i=1}^n a_{\gamma_i}\right) = \boxtimes_{i\neq j} a_{\gamma_i}.$$

6. $\{(X_\gamma, \tau_\gamma) : \gamma \in \Gamma\}$ is *lower projection separated* if $\forall B \subset \Gamma$, $[\forall\beta \in B,\, b_\beta \in \tau_\beta]$ with $\boxplus_{\beta\in B} b_\beta \neq \top$ (in $\bigotimes_{\gamma\in\Gamma} \tau_\gamma$), $\forall\delta \in B$,

$$\left(\pi_\delta\right)_{L\rightarrow} \left(\boxplus_{\beta\in B} b_\beta\right) = b_\delta \quad \text{and} \quad \left(\pi_{\prod_{\beta\neq\delta} X_\beta}\right)_{L\rightarrow} \left(\boxplus_{\beta\in B} b_\beta\right) = \boxplus_{\beta\neq\delta} b_{\beta_i}.$$

5.3.1 Proposition. The conditions of weak prime separation and prime separation are equivalent if the condition of product-separation is assumed.

Separation of factors in L-topological product spaces are closely related to the various normalization properties in L-topology introduced in 2.2.2 above.

5.4 Definition (normalization properties in L-topology).

1. An L-topological space (X, τ) is *normalized* [*co-normalized, binormalized, hypernormalized, conditionally normalized*] if τ is normalized [co-normalized, binormalized, hypernormalized, conditionally normalized, resp.] as in 2.2.2 above.

2. A collection $\{(X_\gamma, \tau_\gamma) : \gamma \in \Gamma\}$ of L-topological spaces is *normalized* [*co-normalized, binormalized, conditionally normalized, hypernormalized*] if $\forall \gamma \in \Gamma$, (X_γ, τ_γ) is normalized [co-normalized, binormalized, conditionally normalized, hypernormalized, resp.].

5.4.1 Lemma. Let $\{(X_\gamma, \tau_\gamma) : \gamma \in \Gamma\}$ be a nonempty collection of nonempty L-topological spaces, $\mathfrak{B}$ be the standard family of basic open sets, and $\mathfrak{B}_{co}$ be the corresponding family of cross-sums.

1. For each basic open set $\boxtimes_{i=1}^n u_{\gamma_i}$,
$$||\boxtimes_{i=1}^n u_{\gamma_i}|| = \bigwedge_{i=1}^n ||u_{\gamma_i}|| \, , \; ||\boxtimes_{i=1}^n u_{\gamma_i}||_{co} = \bigwedge_{i=1}^n ||u_{\gamma_i}||_{co} \, ,$$
for each cross-sum $\boxplus_{i=1}^n u_{\gamma_i}$,
$$||\boxplus_{i=1}^n u_{\gamma_i}|| = \bigvee_{i=1}^n ||u_{\gamma_i}|| \, .$$
and when L is a coframe,
$$||\boxplus_{i=1}^n u_{\gamma_i}||_{co} = \bigvee_{i=1}^n ||u_{\gamma_i}||_{co} \, .$$

2. For each $\gamma \in \Gamma$, $\forall u_\gamma \in \tau_\gamma$,
$$\left|\left|\pi_\gamma^{\leftarrow}(u_\gamma)\right|\right| = ||u_\gamma|| \, , \; \left|\left|\pi_\gamma^{\leftarrow}(u_\gamma)\right|\right|_{co} = ||u_\gamma||_{co} \, .$$

3. $\{(X_\gamma, \tau_\gamma) : \gamma \in \Gamma\}$ is normalized if and only if $\mathfrak{B}$ is normalized if and only if $\mathfrak{B}$ is conormalized.

4. $\{(X_\gamma, \tau_\gamma) : \gamma \in \Gamma\}$ is normalized if and only if $\mathfrak{B}_{co}$ is normalized; and if L is a coframe, then $\{(X_\gamma, \tau_\gamma) : \gamma \in \Gamma\}$ is conormalized if and only if $\mathfrak{B}_{co}$ is conormalized

5. If L is a diframe, then $\{(X_\gamma, \tau_\gamma) : \gamma \in \Gamma\}$ is binormalized if and only if $\mathfrak{B}$ is binormalized if and only if $\mathfrak{B}_{co}$ is binormalized.

6. $\{(X_\gamma, \tau_\gamma) : \gamma \in \Gamma\}$ is hypernormalized if and only if $\mathfrak{B}$ is hypernormalized (AC if Γ nonfinite on necessity).

Proof. *Ad (1).* We note in the cross-product case for norms that the first infinite distributive law yields
$$\begin{aligned} \left|\left|\bigwedge_{i=1}^n \pi_{\gamma_i}^{\leftarrow}(u_{\gamma_i})\right|\right| &= \bigvee_{\langle x_\gamma \rangle \in \prod_{\gamma \in \Gamma} X_\gamma} \left(\bigwedge_{i=1}^n \pi_{\gamma_i}^{\leftarrow}(u_{\gamma_i})\right)(\langle x_\gamma \rangle) \\ &= \bigwedge_{i=1}^n \left[\bigvee_{x_{\gamma_i} \in X_{\gamma_i}} u_{\gamma_i}(x_{\gamma_i})\right] = \bigwedge_{i=1}^n ||u_{\gamma_i}|| \, , \end{aligned}$$
where the independence of variables justifies the distribution of the join across all the meets in the second equals sign; and the cross-product case for conorms follows by associativity of meets. Further, the cross-sum case for norms follows by the associativity for joins, and the cross-sum case for conorms follows dually to the cross-product case for norms using the second infinite distributive law.

Ad (2). We note for norms that for each $u_\beta \in \tau_\beta$,
$$\left|\left|\pi_\beta^{\leftarrow}(u_\beta)\right|\right| = \bigvee_{\langle x_\gamma \rangle \in \prod_{\gamma \in \Gamma} X_\gamma} \left(\pi_\beta^{\leftarrow}(u_\beta)\right) \langle x_\gamma \rangle = \bigvee_{x_\beta \in X_\beta} u_\beta(x_\beta) = ||u_\beta|| \, ,$$
and the connorm case follows dually.

Ad (3,4,5). That the family of spaces has a property implies that the indicated family of open L-subsets of the product has a property follows from the appropriate statement of (1); and the converse directions follow from (2).

Ad (6). Assume $\{(X_\gamma, \tau_\gamma) : \gamma \in \Gamma\}$ is hypernormalized. Let two basic open sets u, v be given. W.L.O.G., these sets may be written with a common subset of indices from Γ, namely,
$$u = \bigwedge_{i=1}^n \pi_{\gamma_i}^{\leftarrow}(u_{\gamma_i}) \, , \quad v = \bigwedge_{i=1}^n \pi_{\gamma_i}^{\leftarrow}(v_{\gamma_i}) \, .$$

Assume $u \nleq v$. This means that $\exists\, j \in \{1, ..., n\}$ with $u_{\gamma_j} \nleq v_{\gamma_j}$; and since $\left(X_{\gamma_j}, \tau_{\gamma_j}\right)$ is hypernormalized, $\exists\, s_{\gamma_j} \in X_{\gamma_i}$ with $u_{\gamma_j}\left(s_{\gamma_j}\right) = \top$ and $v_{\gamma_j}\left(s_{\gamma_j}\right) = \bot$. Since that hypernormalized implies normalized (2.2.3), it is the case that $\forall k \in \{1, ..., n\} - \{j\}$, $\exists\, s_{\gamma_k} \in X_{\gamma_k}$ with $u_{\gamma_k}\left(s_{\gamma_k}\right) = \top$. Now choose

$$\langle z_\gamma \rangle_{\gamma \in \Gamma - \{\gamma_1, ..., \gamma_n\}} \in \prod_{\gamma \in \Gamma - \{\gamma_1, ..., \gamma_n\}} X_\gamma,$$

and then put $\langle x_\gamma \rangle \in \prod_{\gamma \in \Gamma} X_\gamma$ by

$$x_\gamma = \begin{cases} s_\gamma, \ \gamma = \gamma_1, ..., \gamma_n, \\ z_\gamma, \ \gamma \neq \gamma_1, ..., \gamma_n. \end{cases}.$$

It follows that

$$\begin{aligned} u\langle x_\gamma \rangle &= \left(\bigwedge_{i=1}^n \pi_{\gamma_i}^{\leftarrow}\left(u_{\gamma_i}\right)\right)\langle x_\gamma \rangle = \bigwedge_{i=1}^n u_{\gamma_i}\left(s_{\gamma_i}\right) = \bigwedge_{k=1}^n \top = \top, \\ v\langle x_\gamma \rangle &= \left(\bigwedge_{i=1}^n \pi_{\gamma_i}^{\leftarrow}\left(v_{\gamma_i}\right)\right)\langle x_\gamma \rangle = v_{\gamma_j}\left(s_{\gamma_i}\right) \wedge \left(\bigwedge_{k \neq j} v_{\gamma_k}\left(s_{\gamma_k}\right)\right) \\ &= \bot \wedge \left(\bigwedge_{k \neq j} v_{\gamma_k}\left(s_{\gamma_k}\right)\right) = \bot, \end{aligned}$$

which concludes the proof of necessity. Sufficiency follows immediately from the identity $\left(\pi_\beta^{\leftarrow}\left(u_\beta\right)\right)\langle x_\gamma \rangle = u_\beta\left(x_\beta\right)$. $\square$

5.5 Lemma (normalization and separation conditions). Let $\{(X_\gamma, \tau_\gamma) : \gamma \in \Gamma\}$ be a collection of nonempty L-topological spaces with $|\Gamma| \geq 2$. Then the following hold:

1. $\{(X_\gamma, \tau_\gamma) : \gamma \in \Gamma\}$ is normalized if and only if it is upper projection separated, in which case it is product separated.
2. If L is a coframe, then $\{(X_\gamma, \tau_\gamma) : \gamma \in \Gamma\}$ is conormalized if and only it is lower projection separated, in which case it is sum separated.
3. If $\{(X_\gamma, \tau_\gamma) : \gamma \in \Gamma\}$ is hypernormalized, then it is prime separated (AC if Γ nonfinite).
4. Converses in (3) and the latter parts of (1,2) fail to hold if $|L| \geq 3$.

Proof. *Ad (1).* For necessity, we assume $\{(X_\gamma, \tau_\gamma) : \gamma \in \Gamma\}$ is normalized and let $n \in \mathbb{N}$, $\{\gamma_i\}_{i=1}^n \subset \Gamma$, $(\forall i = 1, ..., n,\ a_{\gamma_i} \in \tau_{\gamma_i})$ with $\boxtimes_{i=1}^n a_{\gamma_i} \neq \bot$, and $j = 1, ..., n$. Then using Lemma 5.4.1, it follows

$$\begin{aligned} \left(\pi_{\gamma_j}^{\rightarrow}\left(\boxtimes_{i=1}^n a_{\gamma_i}\right)\right)\left(x_{\gamma_j}\right) &= \bigvee_{\pi_{\gamma_j}\langle y_\gamma \rangle_{\gamma \in \Gamma} = x_{\gamma_j}} \left(\boxtimes_{i=1}^n a_{\gamma_i}\right)\langle y_\gamma \rangle_{\gamma \in \Gamma} \\ &= a_{\gamma_j}\left(x_{\gamma_j}\right) \wedge \bigvee_{\langle y_{\gamma_i} \rangle_{i \neq j}} \left(\boxtimes_{i \neq j} a_{\gamma_i}\right)\langle y_{\gamma_i} \rangle \\ &= a_{\gamma_j}\left(x_{\gamma_j}\right) \wedge ||\boxtimes_{i \neq j} a_{\gamma_i}|| \\ &= a_{\gamma_j}\left(x_{\gamma_j}\right) \wedge \bigwedge_{i \neq j} ||a_{\gamma_i}|| \\ &= a_{\gamma_j}\left(x_{\gamma_j}\right) \wedge \top = a_{\gamma_j}\left(x_{\gamma_j}\right), \end{aligned}$$

where the condition $\boxtimes_{i=1}^n a_{\gamma_i} \neq \bot$ implies that each $a_{\gamma_i} \neq \bot$ and hence $||a_{\gamma_i}|| = \top$; and it also follows that

$$\begin{aligned} \left(\pi_{\prod_{\gamma \neq \gamma_j} X_\gamma}^{\rightarrow}\left(\boxtimes_{i=1}^n a_{\gamma_i}\right)\right)\langle x_{\gamma_i} \rangle_{i \neq j} &= \bigvee_{\pi_{\prod_{\gamma \neq \gamma_j} X_\gamma}\langle y_\gamma \rangle_{\gamma \in \Gamma} = \langle x_{\gamma_i} \rangle_{i \neq j}} \left(\boxtimes_{i=1}^n a_{\gamma_i}\right)\langle y_\gamma \rangle_{\gamma \in \Gamma} \\ &= \left(\boxtimes_{i \neq j} a_{\gamma_i}\right)\langle x_{\gamma_i} \rangle_{i \neq j} \wedge \bigvee a_{\gamma_j}\left(x_{\gamma_j}\right) \\ &= \left(\boxtimes_{i \neq j} a_{\gamma_i}\right)\langle x_{\gamma_i} \rangle_{i \neq j} \wedge \left|\left|a_{\gamma_j}\right|\right| \\ &= \left(\boxtimes_{i \neq j} a_{\gamma_i}\right)\langle x_{\gamma_i} \rangle_{i \neq j} \wedge \top \\ &= \left(\boxtimes_{i \neq j} a_{\gamma_i}\right)\langle x_{\gamma_i} \rangle_{i \neq j}, \end{aligned}$$

where the condition $\boxtimes_{i=1}^{n} a_{\gamma_i} \neq \underline{\perp}$ implies $\left|\left|a_{\gamma_j}\right|\right| = \top$. So we now have that $\{(X_\gamma, \tau_\gamma) : \gamma \in \Gamma\}$ is upper projection separated. Now for sufficiency, assume $n \geq 2$, let $j = 1, ..., n$ and $a_{\gamma_j} \in \tau_{\gamma_j}$ be arbitrary, and choose each a_{γ_i} for $i \neq j$ to be $\underline{\top}$. Then from the assumption that $\{(X_\gamma, \tau_\gamma) : \gamma \in \Gamma\}$ is upper projection separated and the computation just above, it follows that

$$\begin{aligned} \underline{\top} &= (\boxtimes_{i \neq j} a_{\gamma_i}) = \pi^{\rightarrow}_{\prod_{\gamma \neq \gamma_j} X_\gamma} (\boxtimes_{i=1}^{n} a_{\gamma_i}) \\ &= (\boxtimes_{i \neq j} a_{\gamma_i}) \wedge \underline{\left|\left|a_{\gamma_j}\right|\right|} = \underline{\top} \wedge \underline{\left|\left|a_{\gamma_j}\right|\right|}, \end{aligned}$$

and hence that $\left|\left|a_{\gamma_j}\right|\right| = \top$. Hence each $\left(X_{\gamma_j}, \tau_{\gamma_j}\right)$ is normalized; and so $\{(X_\gamma, \tau_\gamma) : \gamma \in \Gamma\}$ is normalized.

Now to see that $\{(X_\gamma, \tau_\gamma) : \gamma \in \Gamma\}$ is product separated, let $n \in \mathbb{N}$, $\{\gamma_i\}_{i=1}^{n} \subset \Gamma$, $(\forall i = 1, ..., n,\ a_{\gamma_i}, b_{\gamma_i} \in \tau_{\gamma_i})$ with $\boxtimes_{i=1}^{n} a_{\gamma_i} \neq \underline{\perp}$ and assume $\boxtimes_{i=1}^{n} a_{\gamma_i} \leq \boxtimes_{i=1}^{n} b_{\gamma_i}$. Let $j = 1, ..., n$ be arbitrary. Then the upper projection separation of $\{(X_\gamma, \tau_\gamma) : \gamma \in \Gamma\}$ and the isotonicity of $\left(\pi_{\gamma_j}\right)^{\rightarrow}_{L}$ trivially yield the following:

$$a_{\gamma_j} = \pi^{\rightarrow}_{\gamma_j} (\boxtimes_{i=1}^{n} a_{\gamma_i}) \leq \pi^{\rightarrow}_{\gamma_j} (\boxtimes_{i=1}^{n} b_{\gamma_i}) = b_{\gamma_j}.$$

It follows that $\{(X_\gamma, \tau_\gamma) : \gamma \in \Gamma\}$ is product separated, completing the proof of (1).

Ad (2). The proofs are dual to those for (1).

Ad (3). Assume $\{(X_\gamma, \tau_\gamma) : \gamma \in \Gamma\}$ is a hypernormalized family of nonempty spaces, and let all of the following: $n \in \mathbb{N}, \{\gamma_i\}_{i=1}^{n} \subset \Gamma$, $a_{\gamma_i} \in \tau_{\gamma_i}$, $B \subset \Gamma$ with $\{\gamma_i\}_{i=1}^{n} \subset B$, $b_\beta \in \tau_\beta - \{\underline{\top}\}$. Since each (X_γ, τ_γ) is hypernormalized, 2.2.3 applies to say that $\forall \beta \in B$, $\exists\, y_\beta \in X_\beta$, $b_\beta (y_\beta) = \perp$. Also $\forall \gamma \in \Gamma - B$, let $y_\gamma \in X_\gamma$. It follows that

$$(\boxplus_{\beta \in B} b_\beta) \langle y_\gamma \rangle_{\gamma \in \Gamma} = \bigvee_{\beta \in B} b_{\beta_i} (y_\beta) = \bigvee_{\beta \in B} \perp = \perp.$$

Hence $\boxplus_{\beta \in B} b_\beta \neq \underline{\top}$. Now assume $\boxtimes_{i=1}^{n} a_{\gamma_i} \leq \boxplus_{\beta \in B} b_\beta$; and let us further assume that

$$\forall\, i = 1, ..., n,\ a_{\gamma_i} \nleq b_{\gamma_i}.$$

Then the hypernormalized condition implies

$$\forall\, i = 1, ..., n, \exists\, z_{\gamma_i} \in X_{\gamma_i},\ a_{\gamma_i} (z_{\gamma_i}) = \top,\ b_{\gamma_i} (z_{\gamma_i}) = \perp.$$

Choose $\langle x_\gamma \rangle_{\gamma \in \Gamma}$ so that

$$\begin{aligned} \forall\, i &= 1, ..., n,\ x_{\gamma_i} = z_{\gamma_i}, \\ \forall \gamma &\in \Gamma - \{\gamma_i\}_{i=1}^{n},\ x_\gamma = y_\gamma \text{ from above.} \end{aligned}$$

It follows, using the associativity of joins, that

$$\begin{aligned} (\boxtimes_{i=1}^{n} a_{\gamma_i}) \langle x_\gamma \rangle_{\gamma \in \Gamma} &= \bigwedge_{i=1}^{n} a_{\gamma_i} (x_{\gamma_i}) \\ &= \bigwedge_{i=1}^{n} \top = \top \\ &\nleq \perp = \bigvee_{\beta \in B} \perp \\ &= \left(\bigvee_{i=1}^{n} b_{\gamma_i} (x_{\gamma_i}) \right) \vee \left(\bigvee_{\beta \in B - \{\gamma_i\}_{i=1}^{n}} b_{\beta_i} (x_\beta) \right) \\ &= (\boxplus_{\beta \in B} b_\beta) \langle x_\gamma \rangle_{\gamma \in \Gamma}, \end{aligned}$$

a contradiction. Hence $\{(X_\gamma, \tau_\gamma) : \gamma \in \Gamma\}$ is prime separated, completing the proof of (3) and the lemma. □

5.5.1 Corollary. If a family of spaces is hypernormalized, then it possess all the conditions occurring in 5.3, 5.4, 5.5.

5.6 Lemma (existence of prime open subsets in L-product topology). Let $\{(X_\gamma, \tau_\gamma) : \gamma \in \Gamma\}$ be prime separated and $|L| \geq 2$. Then arbitrary cross sums of prime L-open subsets are prime L-open subsets in the L-product topology. This holds in particular if $\{(X_\gamma, \tau_\gamma) : \gamma \in \Gamma\}$ is hypernormalized (AC if Γ nonfinite).

Proof. Let $B \subset \Gamma$; and $\forall \beta \in \beta$, let $b_\beta \in \Pr(\tau_\beta)$. To show that $\boxplus_{\beta \in B} b_\beta \in \Pr\left(\bigotimes_{\gamma \in \Gamma} \tau_\gamma\right)$, we first note by the prime separation condition that $\boxplus_{\beta \in B} b_\beta \neq \underline{\top}$. Now let $u, v \in \bigotimes_{\gamma \in \Gamma} \tau_\gamma$ such that

$$u \wedge v \leq \bigotimes_{\gamma \in \Gamma} \tau_\gamma.$$

It is to be shown that either $u \leq \boxplus_{\beta \in B} b_\beta$ or $v \leq \boxplus_{\beta \in B} b_\beta$. We assume the contrary, namely we assume that

$$u \nleq \boxplus_{\beta \in B} b_\beta, \ v \nleq \boxplus_{\beta \in B} b_\beta.$$

Hence there exist basic L-open subsets, $\exists \boxtimes_{i=1}^{n_1} u_{\delta_i}, \boxtimes_{j=1}^{n_2} v_{\zeta_j}$ satisfying the following constraints:

$$\boxtimes_{i=1}^{n_1} u_{\delta_i} \leq u, \ \boxtimes_{j=1}^{n_2} v_{\zeta_j} \leq v,$$

$$\boxtimes_{i=1}^{n_1} u_{\delta_i} \nleq \boxplus_{\beta \in B} b_\beta, \ \boxtimes_{j=1}^{n_2} v_{\zeta_j} \nleq \boxplus_{\beta \in B} b_\beta.$$

$$\left(\boxtimes_{i=1}^{n_1} u_{\delta_i}\right) \wedge \left(\boxtimes_{j=1}^{n_2} v_{\zeta_j}\right) \leq \boxplus_{\beta \in B} b_\beta.$$

By adding finite number of $\underline{\top}$ factors to either $\boxtimes_{i=1}^{n_1} u_{\delta_i}$ or $\boxtimes_{j=1}^{n_2} v_{\zeta_j}$ or both as needed and reindexing, the u_{δ_i}s and v_{ζ_j}s can be given a common (finite) set of indices; and so W.L.O.G. we may rewrite $\boxtimes_{i=1}^{n_1} u_{\delta_i}$, $\boxtimes_{j=1}^{n_2} v_{\zeta_j}$ as $\boxtimes_{k=1}^{m} u_{\kappa_k}$, $\boxtimes_{k=1}^{m} v_{\kappa_k}$, respectively. By the associativity of $\wedge$, it follows that

$$\left(\boxtimes_{k=1}^{m} u_{\kappa_k}\right) \wedge \left(\boxtimes_{k=1}^{m} v_{\kappa_k}\right) = \boxtimes_{k=1}^{m} \left(u_{\kappa_k} \wedge v_{\kappa_k}\right),$$

and so we have that

$$\boxtimes_{k=1}^{m} \left(u_{\kappa_k} \wedge v_{\kappa_k}\right) \leq \boxplus_{\beta \in B} b_\beta.$$

Since $\{(X_\gamma, \tau_\gamma) : \gamma \in \Gamma\}$ is prime separated, it follows that $\exists\, q = 1, ..., m$ such that

$$u_{\kappa_q} \wedge v_{\kappa_q} \leq b_{\kappa_q},$$

where u_{κ_q} is one of the original u_{δ_i}s, v_{κ_q} is one of the original v_{ζ_j}s, and b_{κ_q} is one of the original b_βs. Now b_{κ_q} is prime, and so either $u_{\kappa_q} \leq b_{\kappa_q}$ or $v_{\kappa_q} \leq b_{\kappa_q}$; W.L.O.G., say, $u_{\kappa_q} \leq b_{\kappa_q}$. Let a point $\langle x_\gamma \rangle_{\gamma \in \Gamma} \in \prod_{\gamma \in \Gamma} X_\gamma$ be given. Then

$$\begin{aligned} \left(\boxtimes_{i=1}^{n_1} u_{\delta_i}\right) \langle x_\gamma \rangle_{\gamma \in \Gamma} &= \bigwedge_{i=1}^{n_1} u_{\delta_i}\left(x_{\delta_i}\right) \leq u_{\kappa_q}\left(x_{\kappa_q}\right) \\ &\leq b_{\kappa_q}\left(x_{\kappa_q}\right) \leq \bigvee_{\beta \in B} b_\beta\left(x_\beta\right) \\ &= \left(\boxplus_{\beta \in B} b_\beta\right) \langle x_\gamma \rangle_{\gamma \in \Gamma}, \end{aligned}$$

a contradiction to the contrary assumption made above. Hence $\boxplus_{\beta \in B} b_\beta \in \Pr\left(\bigotimes_{\gamma \in \Gamma} \tau_\gamma\right)$. The second assertion follows from 5.5. $\square$

Lemma 5.6 justifies the terminology "prime separation" in 5.3(3) and subsequently throughout the paper. It should be noted that under certain conditions, 5.6 has a converse so that prime L-open sets in the products are precisely the cross sums of prime L-open subsets—see 7.2.2 below.

5.7 Example Classes for Separation and Normalization Conditions. Since hypernormalized is the strongest normalization condition considered in this paper and implies all the separation conditions, then it suffices to build classes of hypernormalized classes; and hence our attention is primarily on these kinds of L-topological spaces.

5.7.1 Example Class Generated by G_χ (Subsection 2.4). All the L-topological spaces in $G_\chi^{\rightarrow}\,|\mathbf{Top}|$ are hypernormalized; equivalently, all the spaces of **2-Top** are hypernormalized.

5.7.2 Example Class Preserved by $LPT \circ L\Omega : L\text{-}\mathbf{Top} \to L\text{-}\mathbf{SobTop}$ (Section 4). The L-soberification functor $LPT \circ L\Omega$ preserves normalized, conormalized, binormalized, and hypernormalized L-topological spaces.

Proof. To see that L-soberification preserves norms, let (X, τ) be an L-topological space and let $u \in \tau$. Then it follows that

$$\begin{aligned} ||\Phi_L(u)|| &= \bigvee_{p \in Lpt(\tau)} \Phi_L(u)(p) \\ &= \bigvee_{p \in Lpt(\tau)} p(u) \\ &\geq \bigvee_{x \in X} \Psi_L(x)(u) \\ &= \bigvee_{x \in X} u(x) \\ &= ||u||, \end{aligned}$$

where we use the fact that $(\Psi_L)^{\rightarrow}(X) \subset Lpt(\tau)$. Hence L-soberification preserves normalized spaces. A similar manipulation shows that L-soberification preserves conorms and hence conormalized spaces; and so L-soberification preserves binormalized spaces.

To see that hpernormalized spaces are preserved by L-soberification, let (X, τ) be hypernormalized and suppose $b, d \in \tau$ with $\Phi_L(b) \nleq \Phi_L(d)$. Since Φ_L is isotone, it follows that $b \nleq d$ and hence that $\exists x_0 \in X$, $b(x_0) = \top$, $d(x_0) = \bot$. It follows that

$$\begin{aligned} \top &= b(x_0) = \Psi_L(x_0)(b) = \Phi_L(b)(\Psi_L(x_0)), \\ \bot &= d(x_0) = \Psi_L(x_0)(d) = \Phi_L(d)(\Psi_L(x_0)). \end{aligned}$$

Hence, $\exists p \in \mathrm{Lpt}(\tau)$, namely $p = \Psi_L(x_0)$, such that $\Phi_L(b)(p) = \top$ and $\Phi_L(d)(p) = \bot$. Hence the L-spectrum $LPT(L\Omega(X, \tau))$ is hypernormalized. $\square$

5.7.3 Example Class Generated by $LPT \circ \Omega$: **Top** $\rightarrow$ L-**SobTop** (Section 4). All the L-topological spaces in $(LPT \circ \Omega)^{\rightarrow} |\mathbf{Top}|$ are hypernormalized.

Proof. Since the L-**2** soberification functor is isomorphic to the functor $LPT \circ L\Omega \circ G_{\chi}$, the claim follows from 5.7.1 and 5.7.2. $\square$

5.7.4 Example Class of Fuzzy Real Lines and Fuzzy Unit Intervals. Each of the alternative fuzzy real line $\mathbb{R}^*(L)$ and alternative fuzzy unit interval $\mathbb{I}^*(L)$ are hypernormalized; and if L is a complete Boolean algebra, the standard fuzzy real line $\mathbb{R}(L)$ and standard fuzzy unit interval $\mathbb{I}(L)$ are hypernormalized.

Proof. Since $\mathbb{R}^*(L) = LPT(\mathbb{R}, \mathfrak{T})$, $\mathbb{I}^*(L) = LPT(\mathbb{R}, \mathfrak{T}_{\mathbb{I}})$, where $\mathfrak{T}$ is the standard topology on $\mathbb{R}$ and $\mathfrak{T}_{\mathbb{I}}$ is the subspace topology on $\mathbb{I} = [0, 1]$, the claims about $\mathbb{R}^*(L)$ and $\mathbb{I}^*(L)$ are an immediate consequence of 5.7.3. Now the claims for $\mathbb{R}(L)$ and $\mathbb{I}(L)$ follow from those for $\mathbb{R}^*(L)$ and $\mathbb{I}^*(L)$ in this way: first, from Theorem 2.16.5(2,3) [24], we have (for L a complete Boolean algebra) that $\mathbb{R}^*(L)$ is L-homeomorphic to $\mathbb{R}(L)$ and $\mathbb{I}^*(L)$ is L-homeomorphic to $\mathbb{I}(L)$; and, second, from 2.2.4(2,3) above, it is easy to show that the L-homeomorph of a hypernormalized L-topological space is hypernormalized. $\square$

5.7.5 Example Class of Non-Generated Examples. We now construct typical examples of binormalized and hypernormalized spaces which are not generated by any of the functors G_{χ}, $LPT \circ L\Omega$, $LPT \circ \Omega$.

1. **Non-generated hypernormalized space**. Let $X = \{x_1, x_2, x_3\}$, $L \cong \mathbb{B}_4 = \{\bot, \alpha, \beta, \top\}$, with $\alpha \wedge \beta = \bot$, $\alpha \vee \beta = \top$, and $\tau = \{\underline{\bot}, u, \underline{\top}\}$, where $u = \langle \alpha, \top, \bot \rangle$, i.e.,

$$u(x_1) = \alpha,\ u(x_2) = \top,\ u(x_3) = \bot.$$

The following claims hold:

(a) *The L-topological space (X, τ) is hypernormalized.* This is clear by inspection.

(b) $(X, \tau) \notin G_{\chi}^{\rightarrow} \mid \mathbf{Top}|$. This is clear by inspection (since u takes a value other than $\bot$ or $\top$).

(c) $(X, \tau) \notin LPT^{\rightarrow} |L\text{-}\mathbf{Top}|$. Since each output space of LPT is L-sober (4.2(8) above), it suffices to show that (X, τ) is not L-sober; and for this, it suffices to show that Ψ_L is not surjective (4.1 above). Now τ is a chain of 3 elements, and from this we can see that $Lpt(\tau)$ has 4 frame maps: the first map takes τ onto the "left-side" subchain of $\mathbb{B}_4$ ($u \mapsto \alpha$), the second takes τ onto the "right-side" subchain of $\mathbb{B}_4$ ($u \mapsto \beta$), the third map takes u and $\underline{\top}$ to $\top$, and the fourth map takes u and $\underline{\perp}$ to $\perp$. But X has only 3 elements, so Ψ_L cannot be surjective (though we observe that Ψ_L is injective, i.e., (X,τ) is L-T_0, since u separates any two points of X from each other). $\square$

2. **Non-generated binormalized space which is not hpernormalized**. Let $X = \{x_1, x_2, x_3, x_4\}$, $L \cong \mathbb{B}_4$,

$$u = \langle \alpha, \top, \top, \perp \rangle, \; v = \langle \beta, \top, \top, \perp \rangle,$$

and $\tau = \{\underline{\perp}, u \wedge v, u, v, u \vee v, \underline{\top}\}$. Then (X,τ) is a binormalized space which is not hypernormalized; clearly $(X,\tau) \notin G_\chi^{\rightarrow} \mid \mathbf{Top}|$; and $(X,\tau) \notin LPT^{\rightarrow} |L\text{-}\mathbf{Top}|$ since (X,τ) is not L-T_0—none of the L-open subsets separate x_2 and x_3. $\square$

5.7.6 Summary of Example Classes. There is a rich inventory of families of generated and non-generated hypernormalized spaces in L-**Top**; these families include historically important examples of L-topological spaces; and by Lemma 5.6, such families exhibit prime-separation and their L-topological products have a sufficiently rich supply of prime (basic) L-open subsets relative to the prime open L-subsets of the underlying factor spaces.

6 Set-Up of Sections 7, 8, 9 and Proof of Theorem 1.6.1

This section sets up the proofs of Theorems 1.7, 1.2, 1.7.1, 1.8, and 1.9, and at the end of the section gives the proof of Theorem 1.6.1.

Throughout this section, we have a family $\{(X_\gamma, \tau_\gamma) : \gamma \in \Gamma\}$ of L-topological spaces, the localic product $\bigoplus_{\gamma\in\Gamma} \tau_\gamma$ is assumed L-spatial, and the goal is to prove that $\bigoplus_{\gamma\in\Gamma} \tau_\gamma \cong \bigotimes_{\gamma\in\Gamma} \tau_\gamma$ (sufficiency of 1.3). Under these assumptions, together with the added assumption that L is M-spatial with M a subframe of L, the last line of the proof of 1.5 in 4.4 above gives

$$\bigoplus_{\gamma\in\Gamma} \tau_\gamma \cong \bigotimes_{\gamma\in\Gamma} (\Phi_M)^{\rightarrow}(\tau_\gamma), \qquad (6.1)$$

where the right-side is an L-product topology. In the claimed proof of [2] for the sufficiency of Theorem 1.2, it is precisely at 6.1 (but with traditional topologies) that the authors state that the proof is trivial and left to the reader. We respectfully disagree: it is precisely at 6.1 that the proof becomes difficult and interesting.

Before analyzing why the proof after 6.1 might have seemed trivial in [2], we describe an easy logical trap, a fallacy which we illustrate using two families $\{(X_1,\tau_1), (X_2,\tau_2)\}, \{(Y_1,\sigma_1), (Y_2,\sigma_2)\}$ of two L-topological spaces each. Suppose that L-topologies of the same index are order-isomorphic, i.e.,

$$\tau_1 \cong \sigma_1, \; \tau_2 \cong \sigma_2.$$

Is it necessarily the case that

$$\tau_1 \otimes \tau_2 \cong \sigma_1 \otimes \sigma_2\,? \qquad (6.2)$$

If we already knew that

$$\tau_1 \otimes \tau_2 \cong \tau_1 \oplus \tau_2, \; \sigma_1 \otimes \sigma_2 \cong \sigma_1 \oplus \sigma_2, \qquad (6.3)$$

then because $\bigoplus$ gives the categorical coproduct in **Frm** (Section 3 above), 6.2 would necessarily follow; but 6.3 is essentially what is to be proved under the assumption that the localic products in question are spatial (or L-spatial as in the case of 1.3)! Restated, 6.2 asserts that the topological product behaves as the localic product with respect to L-topologies that are index-wise order-isomorphic.

Now assuming we have 6.2 (extended to arbitrary indexing sets Γ), then 6.1 does indeed quickly give the proof of 1.2 and 1.3: in the case of 1.3 with the added assumption that L is assumed M-spatial with M a subframe of L, we have from 4.2(3) and 4.2.1(1) that $\forall \gamma \in \Gamma$, $\tau_\gamma \cong (\Phi_M)^{\rightarrow}(\tau_\gamma)$; hence by 6.2 (as extended), we would have

$$\bigotimes_{\gamma\in\Gamma} (\Phi_M)^{\rightarrow}(\tau_\gamma) \cong \bigotimes_{\gamma\in\Gamma} \tau_\gamma, \qquad (6.4)$$

from which sufficiency in 1.3 would immediately follow from 6.1.

Such an easy path is not available to us: we must prove 6.4 (and its traditional counterpart for 1.2) without the benefit of having 6.2 (extended) beforehand.

Since we cannot logically proceed as if both $\bigotimes_{\gamma\in\Gamma}(\Phi_M)^{\rightarrow}(\tau_\gamma)$, $\bigotimes_{\gamma\in\Gamma}\tau_\gamma$ behave as categorical products, i.e., as localic or point-free products, we must somehow take into account the underlying carrier sets of all these L-topologies. Now since $\bigotimes_{\gamma\in\Gamma}\tau_\gamma$ is an L-topology and M-spatial (4.2.1(1)), it follows immediately from 4.2(3) that

$$(\Phi_M)^{\rightarrow}\left(\bigotimes_{\gamma\in\Gamma}\tau_\gamma\right)\cong\bigotimes_{\gamma\in\Gamma}\tau_\gamma. \qquad (6.5)$$

Therefore to obtain 6.4, it suffices by 6.5 to obtain

$$(\Phi_M)^{\rightarrow}\left(\bigotimes_{\gamma\in\Gamma}\tau_\gamma\right)\cong\bigotimes_{\gamma\in\Gamma}(\Phi_M)^{\rightarrow}(\tau_\gamma). \qquad (6.6)$$

We now consider the underlying carrier sets and prove 6.6 (under certain conditions) by applying 2.4.2(3) above after *first* showing that the following two L-topological spaces are L-homeomorphic:

$$\left(Mpt\left(\bigotimes_{\gamma\in\Gamma}\tau_\gamma\right),(\Phi_M)^{\rightarrow}\left(\bigotimes_{\gamma\in\Gamma}\tau_\gamma\right)\right),\quad\left(\prod_{\gamma\in\Gamma}Mpt(\tau_\gamma),\bigotimes_{\gamma\in\Gamma}(\Phi_M)^{\rightarrow}(\tau_\gamma)\right).$$

In effect, we are going to prove (under certain conditions) that the composition $MPT\circ M\Omega$ preserves categorical products, and the difficulty in doing this lies in the fact that $M\Omega$ need not preserve categorical products.

We now construct a ground map

$$f_M\ :\ \left(Mpt\left(\bigotimes_{\gamma\in\Gamma}\tau_\gamma\right),(\Phi_M)^{\rightarrow}\left(\bigotimes_{\gamma\in\Gamma}\tau_\gamma\right)\right)\rightarrow\left(\prod_{\gamma\in\Gamma}Mpt(\tau_\gamma),\bigotimes_{\gamma\in\Gamma}(\Phi_M)^{\rightarrow}(\tau_\gamma)\right). \qquad (6.7)$$

Let $p:\bigotimes_{\gamma\in\Gamma}\tau_\gamma\rightarrow M$ be a frame map, consider the (restricted) L-preimage operators (2.2 above)

$$\pi_\beta^{\leftarrow}:\tau_\beta\rightarrow\bigotimes_{\gamma\in\Gamma}\tau_\gamma$$

of the projections—which are all frame maps from factor topologies to the product topology (i.e., the projections are L-continuous), and put

$$p_\gamma=p\circ(\pi_\gamma)_L^{\leftarrow},\quad f_M(p)=\langle p_\gamma\rangle_{\gamma\in\Gamma}.$$

Clearly f_M is a well-defined mapping. Thus for each subframe M of L for which L is M-spatial, we have such a map f_M.

6.8 Lemma. The map f_M of 6.7 is injective.

Proof. Let $p\neq q$ in $Mpt\left(\bigotimes_{\gamma\in\Gamma}\tau_\gamma\right)$. Then $\exists\, u\in\bigotimes_{\gamma\in\Gamma}\tau_\gamma$, $p(u)\neq q(u)$. Since M is a subframe of L, the first infinite distributive law implies that u is constructed as a join of meets of subbasic L-open subsets. Hence $\exists\,\beta\in\Gamma$, $\exists\, u_\beta\in\tau_\beta$,

$$p_\beta(u_\beta)=p\left((\pi_\gamma)_L^{\leftarrow}(u_\beta)\right)\neq q\left((\pi_\gamma)_L^{\leftarrow}(u_\beta)\right)=q_\beta.$$

It follows that $p_\beta\neq q_\beta$ and

$$f_M(p)=\langle p_\gamma\rangle_{\gamma\in\Gamma}\neq\langle q_\gamma\rangle_{\gamma\in\Gamma}=f_M(q).\quad\square$$

6.9 Lemma. The map f_M of 6.7 is L-continuous.

Proof. To show that f_M is L-continuous, it suffices to show that f_M is "subbasic L-continuous" (Theorem 3.2.6 of [23]), i.e., that the (Zadeh) preimage of each subbasic open L-subset is an open L-subset. To that end, let v be a subbasic open L-subset of $\bigotimes_{\gamma\in\Gamma}(\Phi_M)^{\rightarrow}(\tau_\gamma)$. This means that $\exists\,\beta\in\Gamma$, $\exists\,\hat{v}\in\tau_\beta$,

$$v=(\pi_\beta)_L^{\leftarrow}(\Phi_M(\hat{v})).$$

Now let $p \in Mpt\left(\bigotimes_{\gamma\in\Gamma}\tau_\gamma\right)$. Then

$$\begin{aligned}
(f_M)_L^{\leftarrow}(v)(p) &= (f_M)_L^{\leftarrow}\left((\pi_\beta)_L^{\leftarrow}(\Phi_M(\hat{v}))\right)(p) = \left((\pi_\beta)_L^{\leftarrow}(\Phi_M(\hat{v}))\right)(f_M(p)) \\
&= \left((\pi_\beta)_L^{\leftarrow}(\Phi_M(\hat{v}))\right)\langle p_\gamma\rangle_{\gamma\in\Gamma} = (\Phi_M(\hat{v}))\left(\pi_\beta\langle p_\gamma\rangle_{\gamma\in\Gamma}\right) \\
&= (\Phi_M(\hat{v}))(p_\beta) = p_\beta(\hat{v}) \\
&= \left(p\circ(\pi_\beta)_L^{\leftarrow}\right)(\hat{v}) = p\left((\pi_\beta)_L^{\leftarrow}(\hat{v})\right) \\
&= \Phi_M\left((\pi_\beta)_L^{\leftarrow}(\hat{v})\right)(p),
\end{aligned}$$

which means that

$$(f_M)_L^{\leftarrow}(v) = \Phi_M\left(\pi_\beta^{\leftarrow}(\hat{v})\right).$$

But $(\pi_\beta)_L^{\leftarrow}(\hat{v})$ is a subbasic open L-subset in $\bigotimes_{\gamma\in\Gamma}\tau_\gamma$, so that

$$(f_M)_L^{\leftarrow}(v) = \Phi_L\left((\pi_\beta)_L^{\leftarrow}(\hat{v})\right) \in \Phi_L^{\rightarrow}\left(\bigotimes_{\gamma\in\Gamma}\tau_\gamma\right).$$

Hence f_M is subbasic L-continuous and therefore L-continuous. □

6.10 Lemma. The map f_M of 6.7 is relatively L-open in the sense of 2.4.1(2).

6.10.1 Sublemma. Let a locale A have a subbasis S in the sense that A comprises all joins of finite meets of members of S. Then for each subframe M of L, $(\Phi_M)^{\rightarrow}(S)$ is a subbasis of the L-topology $(\Phi_M)^{\rightarrow}(A)$.
Proof of Sublemma. This is a consequence of Φ_M being a frame map and M being a subframe of L. □

Proof of 6.10. Since f_M is injective (6.8), it suffices by 2.4.2(3) to show that f_M is "subbasic L-open" with respect to its image. Let $u \in (\Phi_M)^{\rightarrow}\left(\bigotimes_{\gamma\in\Gamma}\tau_\gamma\right)$ be subbasic L-open. By 6.10.1, $\exists\beta\in\Gamma$, $\exists u_\beta\in\tau_\beta$ such that

$$u = \Phi_M\left((\pi_\beta)_L^{\leftarrow}(u_\beta)\right).$$

Let $\langle p_\gamma\rangle_{\gamma\in\Gamma} \in \prod_{\gamma\in\Gamma} Mpt(\tau_\gamma)$ such that $p \in Mpt\left(\bigotimes_{\gamma\in\Gamma}\tau_\gamma\right)$ is the unique point with $f_M(p) = \langle p_\gamma\rangle_{\gamma\in\Gamma}$. Then 2.2.1(3) implies

$$((f_M)_L^{\rightarrow}(u))\left(\langle p_\gamma\rangle_{\gamma\in\Gamma}\right) = u(p),$$

It now follows that

$$\begin{aligned}
((f_M)_L^{\rightarrow}(u))(\langle p_\gamma\rangle_{\gamma\in\Gamma}) &= u(p) \\
&= \left(\Phi_M\left((\pi_\beta)_L^{\leftarrow}(u_\beta)\right)\right)(p) \\
&= p\left((\pi_\beta)_L^{\leftarrow}(u_\beta)\right) \\
&= \left(p\circ(\pi_\beta)_L^{\leftarrow}\right)(u_\beta) \\
&= (f_M)_L^{\rightarrow}(p)_\beta(u_\beta) \\
&= p_\beta(u_\beta) \\
&= (\Phi_M(u_\beta))(p_\beta) \\
&= (\Phi_M(u_\beta))\left(\pi_\beta\left(\langle p_\gamma\rangle_{\gamma\in\Gamma}\right)\right) \\
&= (\Phi_M(u_\beta))(\pi_\beta(f_M(p))) \\
&= \left((\pi_\beta)_L^{\leftarrow}(\Phi_M(u_\beta))\right)(f_M(p)).
\end{aligned}$$

Hence,

$$((f_M)_L^{\rightarrow}(u))(f_M(p)) = (\pi_\beta)_L^{\leftarrow}(\Phi_M(u_\beta))(f_M(p)),$$

and so

$$\begin{aligned}
((f_M)_L^{\rightarrow}(u))_{|(f_M)^{\rightarrow}\left(Mpt\left(\bigotimes_{\gamma\in\Gamma}\tau_\gamma\right)\right)} &= (\pi_\beta)_L^{\leftarrow}(\Phi_M(u_\beta))_{|(f_M)^{\rightarrow}\left(Mpt\left(\bigotimes_{\gamma\in\Gamma}\tau_\gamma\right)\right)} \\
&\in \left(\bigotimes_{\gamma\in\Gamma}\Phi_M^{\rightarrow}(\tau_\gamma)\right)_{|(f_M)^{\rightarrow}\left(Mpt\left(\bigotimes_{\gamma\in\Gamma}\tau_\gamma\right)\right)}.
\end{aligned}$$

Therefore f_M is relatively L-open. □

6.11 Lemma (set-up wrap-up). Let $\{(X_\gamma, \tau_\gamma) : \gamma \in \Gamma\}$ be a family of L-topological spaces with $\bigoplus_{\gamma\in\Gamma} \tau_\gamma$ being L-spatial. For each M a subframe of L such that L is M-spatial, the map

$$f_M : \left(Mpt \left(\bigotimes_{\gamma\in\Gamma} \tau_\gamma \right), (\Phi_M)^{\rightarrow} \left(\bigotimes_{\gamma\in\Gamma} \tau_\gamma \right) \right) \rightarrow \left(\prod_{\gamma\in\Gamma} Mpt\,(\tau_\gamma)\,, \bigotimes_{\gamma\in\Gamma} (\Phi_M)^{\rightarrow}(\tau_\gamma) \right)$$

defined between 6.7 and 6.8 above is an L-embedding. Hence, if f_M is surjective, then f_M is an L-homeomorphism and

$$\bigoplus_{\gamma\in\Gamma} \tau_\gamma \cong \bigotimes_{\gamma\in\Gamma} \tau_\gamma.$$

Restated, under the conditions that M is a subframe of L such that L is M-spatial and that f_M is surjective, 1.3 holds: $\bigoplus_{\gamma\in\Gamma} \tau_\gamma \cong \bigotimes_{\gamma\in\Gamma} \tau_\gamma$ if and only if $\bigoplus_{\gamma\in\Gamma} \tau_\gamma$ is L-spatial.

Proof. Conjoin 6.8, 6.9, 6.10, together with 2.4.1(3); and the restatement also uses 1.4 (see 4.3). □

6.12 Proof of Theorem 1.6.1. We satisfy 6.11 above. First, choose $M = L$ (recall 4.2.1(1)), and let

$$\langle p_\gamma \rangle_{\gamma\in\Gamma} \in \prod_{\gamma\in\Gamma} Lpt\,(\tau_\gamma)\,.$$

Then the L-S_0 condition says that $\exists\ \langle x_\gamma \rangle_{\gamma\in\Gamma} \in \prod_{\gamma\in\Gamma} X_\gamma$ such that

$$\forall \gamma \in \Gamma,\ \Psi_L\,(x_\gamma) = p_\gamma.$$

So we set

$$p \equiv \Psi_L \left(\langle x_\gamma \rangle_{\gamma\in\Gamma} \right)$$

in $Lpt \left(\bigotimes_{\gamma\in\Gamma} \tau_\gamma \right)$. Now fix $\beta \in \Gamma$ and $u_\beta \in \tau_\beta$. Then

$$\begin{aligned}
(f_L\,(p))_\beta\,(u_\beta) &= \left(p \circ \pi_\beta^{\leftarrow}\right)(u_\beta) \\
&= p\left(\pi_\beta^{\leftarrow}\,(u_\beta)\right) \\
&= \Psi_L \left(\langle x_\gamma \rangle_{\gamma\in\Gamma} \right) \left(\pi_\beta^{\leftarrow}\,(u_\beta)\right) \\
&= \pi_\beta^{\leftarrow}\,(u_\beta) \left(\langle x_\gamma \rangle_{\gamma\in\Gamma} \right) \\
&= (u_\beta \circ \pi_\beta) \left(\langle x_\gamma \rangle_{\gamma\in\Gamma} \right) \\
&= u_\beta \left(\pi_\beta \left(\langle x_\gamma \rangle_{\gamma\in\Gamma} \right) \right) \\
&= u_\beta\,(x_\beta) \\
&= \Psi_L\,(x_\beta)\,(u_\beta) \\
&= p_\beta\,(u_\beta)\,.
\end{aligned}$$

It follows that $f\,(p) = \langle p_\gamma \rangle_{\gamma\in\Gamma}$ and that f is surjective. □

The L-S_0 axiom in the assumption of 1.6.1 is further weakened to q-L-S_0 in the presence of other assumptions in 1.7.1—see its proof in Section 8 below.

7 Proofs of Theorem 1.7 and Theorem 1.2

This section gives the proof of Theorem 1.7 and derives Theorem 1.2 as a corollary of Theorem 1.7, and then outlines the proof of Theorem 1.2 independently of Theorem 1.7 to show how the proofs of Theorem 1.7 and Theorem 1.2 were discovered beyond the common setup of 1.7, 1.8, 1.9 given in Section 6 based on correspondence with Prof. Johnstone. In the process, this section gives the first complete proof of Theorem 1.2 known to the authors. Finally, the proofs of Section 6 and this section together yield characterizations of prime L-open subsets of certain L-topological product spaces and characterizations of prime open and irreducible closed subsets of traditional product spaces—see 7.2.2 and 7.4.6.

We begin by collecting and applying some notions from [9].

7.1 $Lpt\,(A)$, $Pt\,(A)$, prime open sets, irreducible closed sets

An element $c \neq \top$ of a lattice A is *prime* if $\forall a, b \in A$ with $a \wedge b \leq c$, $a \leq b$ or $b \leq c$; and we put $\Pr(A)$ for the set of all primes of A. If (X, τ) is an L-topological space, $\Pr(\tau)$ comprises the *prime L-open* subsets of (X, τ); if $(X, \mathfrak{T})$ is a topological space, $\Pr(\mathfrak{T})$ comprises the *prime open* subsets of $(X, \mathfrak{T})$; for $(X, \mathfrak{T})$ we have the notion of an irreducible closed subset—a closed subset F of a topological space $(X, \mathfrak{T})$ is *irreducible* if it cannot be written as a union of proper closed subsets of F; and letting $\mathfrak{F}$ be the collection of closed subsets, we write $Irred\,(\mathfrak{F})$ for the family of all irreducible closed subsets.

Recalling the ideas of Section 4 above, we now let $A \in |\mathbf{Loc}|$ and put

$$\varphi_L : Lpt\,(A) \to \Pr(A) \quad \text{by} \quad \varphi_L\,(p) = a_p \equiv \bigvee_{p(a) = \perp} a,$$

$$\psi_L : \Pr(A) \to Lpt\,(A) \quad \text{by} \quad \psi_L\,(a) = p_a,$$

where

$$p_a : A \to \mathbf{2} \hookrightarrow L, \quad \text{where} \quad p_a\,(b) = \begin{cases} \perp, & b \leq a, \\ \top, & \text{otherwise} \end{cases}.$$

Note that $Lpt\,(A)$ has the ordering of mappings induced from the order of L and $\Pr(A)$ has the relative ordering from A. Also, if B is a poset, then B^{op} indicates the poset with B as the carrier set and the dual order.

7.1.1 Proposition. The following hold:

1. $\varphi_L \dashv^{op} \psi_L$, where "$\dashv^{\mathrm{op}}$" indicates that $\Pr(A)^{\mathrm{op}}$ replaces $\Pr(A)$.
2. $\varphi_L \circ \psi_L = id_{\Pr(A)^{op}}$ (the adjunction of (1) is an iso-coreflection).
3. $\left[\forall A \in |\mathbf{Frm}|\,,\ \psi_L \circ \varphi_L = id_{Lpt(A)}\right] \Leftrightarrow L = \mathbf{2}$.
4. $\psi_L \circ \varphi_L = id_{Lpt(A)}$ need not hold.

Proof. *Ad (1,2).* We have φ_L and ψ_L are antitone and

$$(\varphi_L \circ \psi_L)\,(a) = a_{p_a} = \bigvee_{p_a(c) = \perp} c = a.$$

Also,

$$(\psi_L \circ \varphi_L)\,(p) = \psi_L\,(a_p) = p_{a_p}.$$

Now let $a \in A$. First suppose $a \leq a_p$. Then

$$p(a) \leq p(a_p) = \perp = p_{a_p}\,(a)\,.$$

Second, suppose $a \nleq a_p$. Then

$$p\,(a) \leq \top = p_{a_p}\,(a)\,.$$

Thus $p \leq p_{a_p}$. We so far have

$$\varphi_L \circ \psi_L = id_{\Pr(A)^{op}}, \quad \psi_L \circ \varphi_L \geq id_{Lpt(A)}.$$

Replacing $\Pr(A)$ by $\Pr(A)^{\mathrm{op}}$ yields that φ_L and ψ_L are isotone and that the display just above still holds. So $\varphi_L \dashv^{\mathrm{op}} \psi_L$.

Ad (3). Sufficiency follows noting in the proof of (1) that when $a \nleq a_p$, the definition of a_p implies $p\,(a) \neq \perp$; and p being $\mathbf{2}$-valued then forces $p\,(a) = \top = p_{a_p}\,(a)$. For necessity, assume $|L| > 2$. Then $\exists\, \alpha \in L - \{\perp, \top\}$ and $A \equiv \{\perp, \alpha, \top\}$ is a subframe of L. Choose $p : A \to A \hookrightarrow L$ to be id_A. Note $a_p = \perp$ and that

$$p_{a_p} \equiv p_\perp : A \to L \quad \text{by} \quad p_\perp\,(b) = \begin{cases} \perp, & b = \perp, \\ \top, & \text{otherwise} \end{cases}.$$

Clearly $p_{a_p} \neq p$.

Ad (4). Immediate corollary of the necessity of (3). $\Box$

7.1.2 Corollary. The following hold:

1. $\mathrm{Pr}(A)$ is bijective with $Pt\,(A)$, and this bijection is an antitone isomorphism.
2. $\mathrm{Pr}\,(\mathfrak{T})$ is bijective with $Irred\,(\mathfrak{K})$, and this bijection is an antitone isomorphism.
3. $Irred\,(\mathfrak{K})$ is order-isomorphic to $Pt\,(\mathfrak{T})$.

Proof. (1) is a consequence of 7.1.1(1–3) with $L = \mathbf{2}$. Now (2) follows from sending prime open set U to $X - U$ and then sending irreducible closed F back to $X - F$. And (3) follows from (1) and (2). □

7.2 Proof of Theorem 1.7

Necessity of 1.7 has been given by 1.4. As for sufficiency, 6.11 above shows that the proof of sufficiency is finished once the surjectivity of the map

$$f_M : Mpt\left(\bigotimes_{\gamma\in\Gamma}\tau_\gamma\right) \to \prod_{\gamma\in\Gamma} Mpt\,(\tau_\gamma)$$

of 6.7 is established: surjectivity would make f_M an L-homeomorphism, which would then imply

$$\begin{aligned}
\bigoplus_{\gamma\in\Gamma}\tau_\gamma &\cong \bigotimes_{\gamma\in\Gamma}(\Phi_M)^{\rightarrow}(\tau_\gamma) && \text{(recall 6.1)}\\
&\cong (\Phi_M)^{\rightarrow}\left(\bigotimes_{\gamma\in\Gamma}\tau_\gamma\right) && \text{(f_M an L-homemomorphism, 2.4.2(3))}\\
&\cong \bigotimes_{\gamma\in\Gamma}\tau_\gamma && \text{(6.5)}
\end{aligned}$$

under the assumption that L is M-spatial for some M a subframe of L.

The standing assumptions for 1.7 are that the family $\{(X_\gamma, \tau_\gamma) : \gamma \in \Gamma\}$ of L-topological spaces is prime separated and that L is spatial. Since L is $\mathbf{2}$-spatial, we set $M = \mathbf{2}$. To show that f_M is surjective, we let $\langle q_\gamma\rangle_{\gamma\in\Gamma} \in \prod_{\gamma\in\Gamma} Mpt\,(\tau_\gamma)$. We must find $p \in Mpt\left(\bigotimes_{\gamma\in\Gamma}\tau_\gamma\right)$ such $f_M\,(p) = \langle q_\gamma\rangle_{\gamma\in\Gamma}$. Since $M = \mathbf{2}$, 7.1.2 applies to say that $\forall\gamma \in \Gamma$, $\exists\, u_{q_\gamma} \in \mathrm{Pr}\,(\tau_\gamma)\,,\ \Phi_M\,(q_\gamma) = u_{q_\gamma}$. Choosing $B = \Gamma$ and invoking prime separation, it follows from 5.6 that

$$u \equiv \boxplus_{\gamma\in\Gamma} u_{q_\gamma} \in \mathrm{Pr}\left(\bigotimes_{\gamma\in\Gamma}\tau_\gamma\right).$$

Applying 7.1.2 again, $\exists\, p \equiv p_u \in Mpt\left(\bigotimes_{\gamma\in\Gamma}\tau_\gamma\right)$. We claim that $f_M\,(p) = \langle q_\gamma\rangle$. It suffices to show that $\forall\beta \in \Gamma,\ p_\beta \equiv f_M\,(p)_\beta = q_\beta$, so fix $\beta \in \Gamma$. Now it is the case that $\forall w \in \tau_\beta$,

$$p_\beta\,(w) = p\left((\pi_\beta)^{\leftarrow}_L\,(w)\right) = \bot \Leftrightarrow (\pi_\beta)^{\leftarrow}_L\,(w) \le u,$$

$$q_\beta\,(w) = \bot \Leftrightarrow w \le u_{q_\beta}.$$

Let us assume that $p_\beta\,(w) = \bot$, namely, that $(\pi_\beta)^{\leftarrow}_L\,(w) \le u$. Choosing $\{\beta\}$ as the finite subfamily of Γ and putting $w_\delta \equiv w$ for $\delta \in \{\beta\}$, we have that

$$\boxtimes_{\delta\in\{\beta\}} w_\delta = (\pi_\beta)^{\leftarrow}_L\,(w) \le u = \boxplus_{\gamma\in\Gamma} u_{q_\gamma};$$

and now prime separation directly applies via 5.3(3) to say that

$$w \equiv w_\beta \le u_{q_\beta}.$$

It follows that $q_\beta\,(w) = \bot$. On the other hand, assume $q_\beta\,(w) = \bot$, namely that $w \le u_{q_\beta}$, and let $\langle x_\gamma\rangle_{\gamma\in\Gamma} \in \prod_{\gamma\in\Gamma} X_\gamma$. Then

$$\begin{aligned}
\left((\pi_\beta)^{\leftarrow}_L\,(w)\right)\langle x_\gamma\rangle_{\gamma\in\Gamma} &= w\,(x_\beta) \le u_{q_\beta}\,(x_\beta)\\
&\le \bigvee_{\gamma\in\Gamma} u_{q_\gamma}\,(x_\gamma) = \left(\boxplus_{\gamma\in\Gamma} u_{q_\gamma}\right)\langle x_\gamma\rangle_{\gamma\in\Gamma}\\
&= u\,\langle x_\gamma\rangle_{\gamma\in\Gamma}\,.
\end{aligned}$$

It follows that $p_\beta(w) = \bot$. Altogether, we have that $p_\beta(w) = \bot \Leftrightarrow q_\beta(w) = \bot$. Hence $f_M(p) = \langle q_\gamma \rangle_{\gamma\in\Gamma}$, concluding the proof of Theorem 1.7. $\square$

7.2.1 Corollary. Let $\{(X_\gamma, \tau_\gamma) : \gamma \in \Gamma\}$ be a family of prime separated L-topological spaces (AC if Γ nonfinite). Then $\bigoplus_{\gamma\in\Gamma} \tau_\gamma \cong \bigotimes_{\gamma\in\Gamma} \tau_\gamma$ if and only if $\bigoplus_{\gamma\in\Gamma} \tau_\gamma$ is L-spatial under each of the following conditions

1. L is completely distributive.
2. L is order-isomorphic to a traditional powerset.
3. L is finite (and a frame).
4. L is a complete chain (such as $\mathbb{I} = [0,1]$).

Proof. For (1), let L be completely distributive. Then by Exercise 2.30(3) of [4], L is $\mathbb{I}$-spatial in the sense of 4.1 above, where $\mathbb{I} = [0,1]$. But $\mathbb{I}$ is **2**-spatial: if $a < b$, put $p : \mathbb{I} \to \mathbf{2}$ by $p(t) = 0 \Leftrightarrow t \le a$; then p is a frame map separating a, b; and so Φ is injective. It now follows from 4.2(4) and 4.2.1(1) above that L is **2**-spatial; hence Theorem 1.7 can be applied. Now (2) and (3) follow from (1), and the proof of (4) is contained in the proof of (1). $\square$

7.2.2 Corollary (characterization of L-prime open sets). Let $\{(X_\gamma, \tau_\gamma) : \gamma \in \Gamma\}$ be a family of prime separated L-topological spaces (AC if Γ nonfinite) and L be spatial. Then an L-open subset in the L-topological product space $\left(\prod_{\gamma\in\Gamma} X_\gamma, \bigotimes_{\gamma\in\Gamma} \tau_\gamma\right)$ is prime if and only if it is the cross sum of prime L-open subsets from the factor spaces.

Proof. Sufficiency is given by 5.6 above. As for necessity, let $u \in \Pr\left(\bigotimes_{\gamma\in\Gamma} \tau_\gamma\right)$. Then, referring to 7.1.2 and the proof of 1.7 given above, we have these unique determinations: u determines $p \in Mpt\left(\bigotimes_{\gamma\in\Gamma} \tau_\gamma\right)$, p determines $\langle p_\gamma \rangle_{\gamma\in\Gamma} \in \prod_{\gamma\in\Gamma} Mpt(\tau_\gamma)$, and each p_γ determines $u_\gamma \in \Pr(\tau_\gamma)$. Now since f_M is bijective, we also have these unique determinations: p is uniquely determined by $\langle p_\gamma \rangle_{\gamma\in\Gamma}$ and hence as the frame mapping uniquely determined by $\boxplus_{\gamma\in\Gamma} u_\gamma$. This forces $u = \boxplus_{\gamma\in\Gamma} u_\gamma$. $\square$

7.3 Proof of Theorem 1.2

Let $\{(X_\gamma, \mathfrak{T}_\gamma) : \gamma \in \Gamma\}$ be a collection of ordinary topological spaces (AC if Γ nonfinite). We want to show that $\bigoplus_{\gamma\in\Gamma} \mathfrak{T}_\gamma \cong \bigotimes_{\gamma\in\Gamma} \mathfrak{T}_\gamma$ if and only if $\bigoplus_{\gamma\in\Gamma} \mathfrak{T}_\gamma$ is spatial. First, we choose $L = \mathbf{2}$ and apply G_χ at the fibre level to each $\mathfrak{T}_\gamma$ (Subsection 2.4) and note that we now have a family of prime-separated **2**-topological spaces. From 7.2.1(4), we have that $\bigoplus_{\gamma\in\Gamma} G_\chi(\mathfrak{T}_\gamma) \cong \bigotimes_{\gamma\in\Gamma} G_\chi(\mathfrak{T}_\gamma)$ if and only if $\bigoplus_{\gamma\in\Gamma} G_\chi(\mathfrak{T}_\gamma)$ is spatial. Now each

$$G_\chi(\mathfrak{T}_\gamma) \cong \mathfrak{T}_\gamma.$$

Since $\bigoplus$ gives the categorical product in **Loc**, this insures that

$$\bigoplus_{\gamma\in\Gamma} G_\chi(\mathfrak{T}_\gamma) \cong \bigoplus_{\gamma\in\Gamma} \mathfrak{T}_\gamma.$$

On the other hand, G_χ at the functorial level is an isomorphism and preserves products, so at the fibre level,

$$\bigotimes_{\gamma\in\Gamma} G_\chi(\mathfrak{T}_\gamma) \cong \bigotimes_{\gamma\in\Gamma} \mathfrak{T}_\gamma.$$

Theorem 1.2 now follows. $\square$

7.4 Discovering proofs of Theorems 1.2 and 1.7

The proof of Theorem 1.7, particularly the surjectivity of the map f_M, was discovered by first writing a complete proof of Theorem 1.2 and then deciding how to write the generalization of its proof needed for Theorem 1.7; and what is key for the proof of surjectivity in 1.7 is the idea of cross sums of prime open sets. But how were cross sums of prime open sets discovered?

Since we now have a complete proof of 1.2 in 7.3 above and our purpose is simply the story of discovery, this subsection considers two traditional topological spaces $(X_1, \mathfrak{T}_1), (X_2, \mathfrak{T}_2)$ in order to streamline the situation and focus on the ideas. Further, that focus will be narrowed to studying the surjectivity of the map

$$f_M : (Pt(\mathfrak{T}_1 \otimes \mathfrak{T}_2), \Phi^{\rightarrow}(\mathfrak{T}_1 \otimes \mathfrak{T}_2)) \rightarrow (Pt(\mathfrak{T}_1) \times Pt(\mathfrak{T}_1), \Phi^{\rightarrow}(\mathfrak{T}_1) \otimes \Phi^{\rightarrow}(\mathfrak{T}_2)),$$

where $M = \mathbf{2}$. Let $(q_1, q_2) \in Pt(\mathfrak{T}_1) \times Pt(\mathfrak{T}_1)$, and note from 7.1.2 we have uniquely determined prime open sets $V_{q_1} \in \Pr(\mathfrak{T}_1)$, $V_{q_2} \in \Pr(\mathfrak{T}_2)$. It is most natural to consider cross products, but the problem is that the cross product $V_{q_1} \times V_{q_2}$ need not be in $\Pr(\mathfrak{T}_1 \otimes \mathfrak{T}_2)$. However, we also know from 7.1.2 that $X_1 - V_{q_1}$, $X_2 - V_{q_2}$ are irreducible closed subsets in their respective spaces. The next three results, based on a letter to the second author from Prof. Johnstone in 1987, establish that cross products of irreducible closed subsets are irreducible closed subsets of the topological product space, proved by repackaging the notion of irreducible closed subsets and working in the relative product topology. And then we prove Theorem 1.2 (again).

7.4.1 Lemma (relative product topologies). Let $A \subset X_1$, $B \subset X_2$, and let $(\mathfrak{T}_1 \otimes \mathfrak{T}_2)(A \times B)$ be the relative topology from the product space on $A \times B$, $\mathfrak{T}_1(A)$ be the relative topology from $(X_1, \mathfrak{T}_1)$ on A, and $\mathfrak{T}_2(B)$ be the relative topology from $(X_2, \mathfrak{T}_2)$ on B. Then

$$(\mathfrak{T}_1 \otimes \mathfrak{T}_2)(A \times B) = \mathfrak{T}_1(A) \otimes \mathfrak{T}_2(B).$$

Proof. Since cross product and intersection commute, the standard basis of $(\mathfrak{T}_1 \otimes \mathfrak{T}_2)(A \times B)$ is contained in the standard basis of $\mathfrak{T}_1(A) \otimes \mathfrak{T}_2(B)$; and hence

$$(\mathfrak{T}_1 \otimes \mathfrak{T}_2)(A \times B) \subset \mathfrak{T}_1(A) \otimes \mathfrak{T}_2(B).$$

On the other hand, the continuity of the projections insure that the standard subbasis of $\mathfrak{T}_1(A) \otimes \mathfrak{T}_2(B)$ is contained in $(\mathfrak{T}_1 \otimes \mathfrak{T}_2)(A \times B)$; and hence

$$(\mathfrak{T}_1 \otimes \mathfrak{T}_2)(A \times B) \supset \mathfrak{T}_1(A) \otimes \mathfrak{T}_2(B). \quad \square$$

7.4.2. Lemma (irreducible closed subspaces). Let $(X, \mathfrak{T})$ be an ordinary topological space and F be a closed subset of X with relative topology $\mathfrak{T}(F)$. Then F is irreducible closed in $(X, \mathfrak{T})$ if and only if $\forall O, W \in \mathfrak{T}(F)$,

$$O \neq \varnothing,\ W \neq \varnothing \ \Rightarrow\ O \cap W \neq \varnothing.$$

Proof. For necessity, assume F is irreducible closed and let $O, W \in \mathfrak{T}(F)$ with $O \neq \varnothing$, $W \neq \varnothing$. Now if $O = F$, then $W \subset F$ and $\varnothing \neq W = O \cap W$; and similarly if $W = F$. So suppose $O, W \subsetneqq F$ and put

$$F_1 = F - O, \quad F_2 = F - W.$$

Then F_1, F_2 are closed subsets of F, $F_1 \neq \varnothing \neq F_2$, and $F_1 \neq F \neq F_2$. Now suppose $O \cap W = \varnothing$. Then

$$F = F - (O \cap W) = (F - O) \cup (F - W) = F_1 \cup F_2,$$

a contradiction to F being irreducible. For sufficiency, deny irreducibility of F. So $\exists$ nonempty, proper closed subsets F_1, F_2 such that $F = F_1 \cup F_2$. Put $O = F - F_1$, $W = F - F_2$. Then $O, W \in \mathfrak{T}(F)$ with $O \neq \varnothing$, $W \neq \varnothing$. Hence $O \cap W \neq \varnothing$. But

$$\varnothing = F - (F_1 \cup F_2) = (F - F_1) \cap (F - F_2) = O \cap W,$$

a contradiction. $\square$

7.4.3 Lemma (products of irreducible closed subsets). Let A be an irreducible closed subset of $(X_1, \mathfrak{T}_1)$ and B be an irreducible closed subset of $(X_2, \mathfrak{T}_2)$. Then $A \times B$ is an irreducible closed subset of $(X_1 \times X_2, \mathfrak{T}_1 \otimes \mathfrak{T}_2)$.

Proof. Let $O, W \in (\mathfrak{T}_1 \otimes \mathfrak{T}_2)(A \times B)$ with $O \neq \varnothing$, $W \neq \varnothing$. Then $\exists\ (x, y) \in O$, $\exists\ (z, w) \in W$. By 7.4.1, $\exists U_1 \times U_2$, $V_1 \times V_2 \in \mathfrak{T}_1(A) \otimes \mathfrak{T}_2(B)$ such that

$$(x, y) \in U_1 \times U_2 \subset O, \quad (z, w) \in V_1 \times V_2 \subset W.$$

Since each of U_1, U_2, V_1, V_2 is nonempty, the irreducibility of each of A, B along with 7.4.2 implies that

$$U_1 \cap V_1 \neq \varnothing \neq U_2 \cap V_2.$$

Hence $\exists\, \widehat{x} \in U_1 \cap V_1,\ \widehat{y} \in U_2 \cap V_2$, and

$$\begin{array}{rcl} (\widehat{x}, \widehat{y}) & \in & (U_1 \cap V_1) \times (U_2 \cap V_2) \\ & = & (U_1 \times U_2) \cap (V_1 \times V_2) \\ & \subset & O \cap W. \end{array}$$

So $O \cap W \neq \varnothing$; and by 7.4.2, $A \times B$ is an irreducible closed subset of $(X_1 \times X_2, \mathfrak{T}_1 \otimes \mathfrak{T}_2)$. $\square$

7.4.4 Proof of Theorem 1.2. Returning to the paragraph above 7.4.1, we have the irreducible closed subsets $X_1 - V_{q_1}$, $X_2 - V_{q_2}$. By 7.4.3,

$$(X_1 - V_{q_1}) \times (X_2 - V_{q_2})$$

is an irreducible closed subset of $(X_1 \times X_2, \mathfrak{T}_1 \otimes \mathfrak{T}_2)$. Then by 7.1.2,

$$(X_1 \times X_2) - [(X_1 - V_{q_1}) \times (X_2 - V_{q_2})] \in \Pr(\mathfrak{T}_1 \otimes \mathfrak{T}_2);$$

and then we have

$$p \equiv p_{(X_1 \times X_2) - [(X_1 - V_{q_1}) \times (X_2 - V_{q_2})]} \in Pt(\mathfrak{T}_1 \otimes \mathfrak{T}_2).$$

It can be shown that $f_M(p) = (q_1, q_2)$, making f_M surjective and a homeomorphism, which, via the set-up of Section 6, yields Theorem 1.2. $\square$

7.4.5 Discussion (reconciliation with Subsection 6.3). Generalizing 7.4.4 to the L-valued case is problematic since 7.4.4 uses closed subsets and the full properties of Boolean complementation (both double negation and excluded middle) and L need not be a Boolean algebra. However, we have the simple calculation:

$$\begin{array}{rcl} (X_1 \times X_2) - [(X_1 - V_{q_1}) \times (X_2 - V_{q_2})] & = & \\ (V_{q_1} \times X_2) \cup (X_1 \times V_{q_2}) & = & V_{q_1} + V_{q_2}. \end{array}$$

And thus appears the notion of *cross sums of prime open sets*, a graph of which for two summands can be seen on p. 84 of [28], though that is not the purpose of [28] and the notion of cross sums as prime open sets in a product topology is not given there. Cross sums of prime open sets can be extended to L-valued topology *without Boolean complementation*, this extension requiring the previous sections of this paper. The following corollary of this line of discussion expands 7.2.2 above for traditional product topologies.

7.4.6 Corollary (characterizations of prime open sets and irreducible closed sets). Let $\{(X_\gamma, \mathfrak{T}_\gamma) : \gamma \in \Gamma\} \subset |\mathbf{Top}|$ be a family of traditional topological spaces. The following hold:

1. The prime open subsets of $\left(\prod_{\gamma \in \Gamma} X_\gamma, \bigotimes_{\gamma \in \Gamma} \mathfrak{T}_\gamma\right)$ are precisely the cross sums of prime open subsets of the factor spaces.
2. The irreducible closed subsets of $\left(\prod_{\gamma \in \Gamma} X_\gamma, \bigotimes_{\gamma \in \Gamma} \mathfrak{T}_\gamma\right)$ are precisely the cross products of irreducible closed subsets of the factor spaces.

Proof. From 7.2.2 comes (1) (setting $L = \mathbf{2}$ and invoking G_χ) and from (1) comes (2). $\square$

8 Proof of Theorem 1.7.1

We appeal to Lemma 6.11 under the assumptions that $\{(X_\gamma, \tau_\gamma) : \gamma \in \Gamma\}$ is normalized and prime separated and each space is q-L-S_0. Choosing $M = L$, we are to show that the map

$$f_L : Lpt\left(\bigotimes_{\gamma \in \Gamma} \tau_\gamma\right) \to \prod_{\gamma \in \Gamma} Lpt(\tau_\gamma)$$

is surjective. Let $\langle p_\gamma \rangle_{\gamma \in \Gamma} \in \prod_{\gamma \in \Gamma} Lpt(\tau_\gamma)$. There are four cases: three special cases and then the general case.

8.1 First special case: $\forall \gamma$, $\boldsymbol{coker}(p_\gamma) = \tau_\gamma - \{\underline{\perp}\}$

It follows that each p_γ is $\mathbf{2}$-valued, i.e., that $p_\gamma \in Pt(\tau_\gamma)$. Hence, using prime separation and 7.1.2 as they were used in the proof of Theorem 1.7 (7.2), it can be shown

$$\exists\, p \in Pt\left(\bigotimes_{\gamma\in\Gamma} \tau_\gamma\right) \subset Lpt\left(\bigotimes_{\gamma\in\Gamma} \tau_\gamma\right)$$

such that

$$f_{\mathbf{2}}(p) = \langle p_\gamma \rangle_{\gamma\in\Gamma}.$$

Since $p \in Pt\left(\bigotimes_{\gamma\in\Gamma} \tau_\gamma\right)$, it follows $f_L(p) = f_{\mathbf{2}}(p)$, which completes the proof of this case. $\square$

8.2 Second special case: $\forall \gamma$, $\boldsymbol{coker}(p_\gamma) \neq \tau_\gamma - \{\underline{\perp}\}$

In this case the q-L-S_0 axiom implies that $\forall \gamma \in \Gamma$, $\exists\, x_\gamma \in X_\gamma$, $\Psi_L(x_\gamma) = p_\gamma$. It follows that the method in the proof of Theorem 1.6.1 (4.6) may be employed here to find $p \in LPt\left(\bigotimes_{\gamma\in\Gamma} \tau_\gamma\right)$ such that $f_L(p) = \langle p_\gamma \rangle_{\gamma\in\Gamma}$. $\square$

8.3 Third special case: $|\Gamma| = 2$, $\exists x_1 \in X_1$ with $p_1 = \Psi_L(x_1)$, $\boldsymbol{coker}(p_2) = \tau_2 - \{\underline{\perp}\}$

Let $u \in \tau_1 \otimes \tau_2 - \{\underline{\perp}\}$ have the form

$$u = \bigvee_{\alpha\in A} (u_{\alpha_1} \boxtimes u_{\alpha_2}),$$

where each $u_{\alpha_1} \boxtimes u_{\alpha_2} \neq \underline{\perp}$. Put

$$p(u) = \bigvee_{\alpha\in A} (p_1(u_{\alpha_1}) \wedge p_2(u_{\alpha_2})).$$

Then:

$$\begin{aligned} p(u) &= \bigvee_{\alpha\in A} (\Psi_L(x_1)(u_{\alpha_1}) \wedge \top) \\ &= \bigvee_{\alpha\in A} u_{\alpha_1}(x_1). \end{aligned}$$

It is convenient to note that $p(u) = \bigvee_{\alpha\in A} p_1(u_{\alpha_1})$. Finally, put $p(\underline{\perp}) = \perp$.

We claim that $p : \tau_1 \otimes \tau_2 \to L$ is a well-defined map. Let

$$\bigvee_{\alpha\in A} (u_{\alpha_1} \boxtimes u_{\alpha_2}) = u = \bigvee_{\beta\in B} (v_{\beta_1} \boxtimes v_{\beta_2})$$

as above for $u \in \tau_1 \otimes \tau_2 - \{\underline{\perp}\}$. Now $\{(X_1, \tau_1), (X_2, \tau_2)\}$ being normalized implies that this family is upper projection separated (5.5(1)); and this separation condition, together with the preservation of arbitrary joins by the Zadeh image operator $\pi_1^{\rightarrow}$ of the first projection, now yields

$$\begin{aligned} \bigvee_{\alpha\in A} u_{\alpha_1} &= \bigvee_{\alpha\in A} \pi_1^{\rightarrow}(u_{\alpha_1} \boxtimes u_{\alpha_2}) \\ &= \pi_1^{\rightarrow}\left(\bigvee_{\alpha\in A} (u_{\alpha_1} \boxtimes u_{\alpha_2})\right) \\ &= \pi_1^{\rightarrow}(u) \\ &= \pi_1^{\rightarrow}\left(\bigvee_{\beta\in B} (v_{\beta_1} \boxtimes v_{\beta_2})\right) \\ &= \bigvee_{\beta\in B} \pi_1^{\rightarrow}(v_{\beta_1} \boxtimes v_{\beta_2}) \\ &= \bigvee_{\beta\in B} v_{\beta_1}. \end{aligned}$$

Applying the frame map p_1 gives the following:

$$\begin{aligned} p_1\left(\bigvee_{\alpha\in A} u_{\alpha_1}\right) &= \bigvee_{\alpha\in A} p_1\left(u_{\alpha_1}\right) = \bigvee_{\alpha\in A} \Psi_L\left(x_1\right)\left(u_{\alpha_1}\right) \\ &= \bigvee_{\alpha\in A} u_{\alpha_1}\left(x_1\right) = \bigvee_{\beta\in B} v_{\beta_1}\left(x_1\right) \\ &= \bigvee_{\beta\in B} \Psi_L\left(x_1\right)\left(v_{\beta_1}\right) = \bigvee_{\beta\in B} p_1\left(v_{\beta_1}\right) \\ &= p_1\left(\bigvee_{\beta\in B} v_{\beta_1}\right). \end{aligned}$$

This shows that p is well-defined.

Now to see that p preserves arbitrary nonempty joins, it suffices to let $\{u_\alpha\}_{\alpha\in A} \subset \tau_1 \otimes \tau_2 - \{\underline{\perp}\}$, where each u_α may be written in the form

$$u_\alpha = \bigvee_{\beta\in A_\alpha} \left(u^\alpha_{\beta_1} \boxtimes u^\alpha_{\beta_2}\right)$$

as above. It follows that

$$\begin{aligned} p\left(\bigvee_{\alpha\in A} u_\alpha\right) &= p\left(\bigvee_{\alpha\in A}\bigvee_{\beta\in A_\alpha} \left(u^\alpha_{\beta_1} \boxtimes u^\alpha_{\beta_2}\right)\right) \\ &= p\left(\bigvee_{(\alpha,\beta)\,\in\, A\times\left(\bigcup_{\alpha\in A} A_\alpha\right)} \left(u^\alpha_{\beta_1} \boxtimes u^\alpha_{\beta_2}\right)\right) \\ &= \bigvee_{(\alpha,\beta)\,\in\, A\times\left(\bigcup_{\alpha\in A} A_\alpha\right)} p_1\left(u^\alpha_{\beta_1}\right) \\ &= \bigvee_{\alpha\in A}\bigvee_{\beta\in A_\alpha} p_1\left(u^\alpha_{\beta_1}\right) \\ &= \bigvee_{\alpha\in A} p\left(u_\alpha\right). \end{aligned}$$

Since p satisfies $p\left(\underline{\perp}\right) = \perp$ by definition, we now have that p preserves arbitrary joins.

As for finite meets, the empty case follows since

$$p\left(\underline{\top}\right) = p\left(\underline{\top} \boxtimes \underline{\top}\right) = p_1\left(\underline{\top}\right) = \top.$$

Given that p preserves bottom, it suffices to check the action of p on $u \wedge v$ where $u, v \in \tau_1 \otimes \tau_2 - \{\underline{\perp}\}$ to conclude that p preserves all binary meets. As above, we may write

$$\begin{aligned} u \wedge v &= \bigvee_{\alpha\in A} \left(u_{\alpha_1} \boxtimes u_{\alpha_2}\right) \wedge \bigvee_{\beta\in B} \left(v_{\beta_1} \boxtimes v_{\beta_2}\right) \\ &= \bigvee_{(\alpha,\beta)\in A\times B} \left(\left(u_{\alpha_1} \wedge v_{\beta_1}\right) \boxtimes \left(u_{\alpha_2} \wedge v_{\beta_2}\right)\right), \end{aligned}$$

so that

$$\begin{aligned} p\left(u \wedge v\right) &= \bigvee_{(\alpha,\beta)\in A\times B} p_1\left(u_{\alpha_1} \wedge v_{\beta_1}\right) \\ &= \bigvee_{(\alpha,\beta)\in A\times B} \left(p_1\left(u_{\alpha_1}\right) \wedge p_1\left(v_{\beta_1}\right)\right) \\ &= \bigvee_{\alpha\in A} p_1\left(u_{\alpha_1}\right) \wedge \bigvee_{\beta\in B} p_1\left(v_{\beta_1}\right) \\ &= p\left(u\right) \wedge p\left(v\right). \end{aligned}$$

It now follows that p is a frame map, i.e., that $p \in Lpt\,(\tau_1 \otimes \tau_2)$.

It remains to check that $f_L\,(p) = \langle p_i \rangle_{i=1,2}$. Given $u \in \tau_1$ and $v \in \tau_2$, it follows:

$$(p \circ \pi_1^{\leftarrow})\,(u) = p\,(u \boxtimes \underline{\top}) = p_1\,(u)\,;$$

$$(p \circ \pi_2^{\leftarrow})\,(v) = p\,(\underline{\top} \boxtimes v) = \begin{cases} p\,(\underline{\top}) = \top = p_2\,(v) : v \neq \underline{\bot} \\ p\,(\underline{\bot}) = \bot = p_2\,(v) : v = \underline{\bot} \end{cases} = p_2\,(v)\,.$$

Hence $f_L\,(p) = \langle p_i \rangle_{i=1,2}$.

8.4 General case

We now use the three special cases in 8.1, 8.2, 8.3 above, respectively, to prove Theorem 1.7.1 in full generality. Partition the indexing set Γ into the disjoint union $\Delta \cup E$, where:

$$\forall \delta \in \Delta,\ \exists x_\delta \in X_\delta,\ p_\delta = \Psi_L\,(x_\delta)\,;$$

$$\forall \epsilon \in E,\ coker\,(p_\epsilon) = \tau_\epsilon - \{\underline{\bot}\}\,.$$

If $\Delta = \varnothing$, then this case follows from 8.1 above; and if $E = \varnothing$, then this case follows from 8.2 above. So we assume that each of Δ and E are nonempty. Now for Δ there is a map

$$h_L : Lpt\left(\bigotimes_{\delta \in \Delta} \tau_\delta\right) \to \prod_{\delta \in \Delta} Lpt\,(\tau_\delta)$$

from Section 6 which is surjective by 8.2 above—the prime separation condition needed in that case is inherited by the subfamily $\{(X_\delta, \tau_\delta) : \delta \in \Delta\}$ since cross products and cross sums are finitely associative; and for E there is a map

$$k_L : Lpt\left(\bigotimes_{\epsilon \in E} \tau_\epsilon\right) \to \prod_{\epsilon \in E} Lpt\,(\tau_\epsilon)$$

from Section 6 which is surjective by 8.1 above—the normalization condition needed in that case is trivially inherited by the subfamily $\{(X_\epsilon, \tau_\epsilon) : \epsilon \in E\}$. Further, there is a map

$$g_L : Lpt\left(\bigotimes_{\delta \in \Delta} \tau_\delta \otimes \bigotimes_{\epsilon \in E} \tau_\epsilon\right) \to \prod_{\delta \in \Delta} Lpt\,(\tau_\delta) \times \prod_{\epsilon \in E} Lpt\,(\tau_\epsilon)$$

from Section 6 which is surjective by 8.3 above—viewing $\left(\prod_{\delta \in \Delta} X_\delta, \bigotimes_{\delta \in \Delta} \tau_\delta\right)$ as (X_1, τ_1) and $\left(\prod_{\epsilon \in E} X_\epsilon, \bigotimes_{\epsilon \in E} \tau_\epsilon\right)$ as (X_2, τ_2). Finally, we note that

$$Lpt\left(\bigotimes_{\gamma \in \Gamma} \tau_\gamma\right) = Lpt\left(\bigotimes_{\delta \in \Delta} \tau_\delta \otimes \bigotimes_{\epsilon \in E} \tau_\epsilon\right),$$

$$\prod_{\gamma \in \Gamma} Lpt\,(\tau_\gamma) = \prod_{\delta \in \Delta} Lpt\,(\tau_\delta) \times \prod_{\epsilon \in E} Lpt\,(\tau_\epsilon)$$

and take the associated identity maps

$$id : Lpt\left(\bigotimes_{\gamma \in \Gamma} \tau_\gamma\right) \to Lpt\left(\bigotimes_{\delta \in \Delta} \tau_\delta \otimes \bigotimes_{\epsilon \in E} \tau_\epsilon\right),$$

$$id : \prod_{\gamma \in \Gamma} Lpt\,(\tau_\gamma) \to \prod_{\delta \in \Delta} Lpt\,(\tau_\delta) \times \prod_{\epsilon \in E} Lpt\,(\tau_\epsilon)\,.$$

Altogether, we now have that the map

$$f_L : Lpt\left(\bigotimes_{\gamma \in \Gamma} \tau_\gamma\right) \to \prod_{\gamma \in \Gamma} Lpt\,(\tau_\gamma)$$

may be written as the composition

$$f_L = id \circ (h_L \times k_L) \circ g_L \circ id,$$

and hence that f_L is surjective. This concludes the proof of Theorem 1.7.1 from Lemma 6.11. $\square$

9 Proofs of Theorem 1.8 and Theorem 1.9

Recall that 6.11 gives for each subframe M of L for which L is M-spatial an L-embedding

$$f_M : \left(Mpt\left(\bigotimes_{\gamma \in \Gamma} \tau_\gamma \right), (\Phi_M)^{\rightarrow}\left(\bigotimes_{\gamma \in \Gamma} \tau_\gamma \right) \right) \rightarrow \left(\prod_{\gamma \in \Gamma} Mpt\left(\tau_\gamma\right), \bigotimes_{\gamma \in \Gamma} (\Phi_M)^{\rightarrow}\left(\tau_\gamma\right) \right).$$

The proof of Theorem 1.7 in Subsection 6.2 instantiates 6.11 by choosing $M = \mathbf{2}$ and the proof of Theorem 1.2 instantiates 6.11 by choosing $L = \mathbf{2}$.

To obtain the proof of Theorem 1.8 from 6.11, we must prove the surjectivity of the map f_M under the conditions of Theorem 1.8. To that end, let us first consider $\langle q_\gamma \rangle_{\gamma \in \Gamma} \in \prod_{\gamma \in \Gamma} Mpt\left(\tau_\gamma\right)$ under the conditions of Theorem 1.7. Then the proof of Theorem 1.7 provides $p \in Mpt\left(\bigotimes_{\gamma \in \Gamma} \tau_\gamma \right)$ such that $f_M = \langle q_\gamma \rangle_{\gamma \in \Gamma}$. Now let $u \in \bigotimes_{\gamma \in \Gamma} \tau_\gamma$ and consider the family

$$\mathcal{B}_u = \left\{ \boxtimes_{i=1}^n u_{\gamma_i} : n \in \mathbb{N},\ \{\gamma_i\}_{i=1}^n \subset \Gamma,\ u_{\gamma_i} \in \tau_{\gamma_i},\ \underline{\perp} \neq \boxtimes_{i=1}^n u_{\gamma_i} \leq u \right\}.$$

Since $\mathcal{B}_u$ comprises all basic L-open subsets inside u and p is a frame map, we have

$$\begin{aligned} p\left(u\right) &= p\left(\bigvee \mathcal{B}_u \right) \\ &= \bigvee_{\boxtimes_{i=1}^n u_{\gamma_i} \in \mathcal{B}_u} p\left(\boxtimes_{i=1}^n u_{\gamma_i} \right) \\ &= \bigvee_{\boxtimes_{i=1}^n u_{\gamma_i} \in \mathcal{B}_u} \left[\bigwedge_{i=1}^n q_{\gamma_i}\left(u_{\gamma_i}\right) \right]. \end{aligned} \tag{9.1}$$

Since f_M is a bijection in Theorem 1.7 and every mapping from $\bigotimes_{\gamma \in \Gamma} \tau_\gamma$ to M that is *formally defined* by 9.1 satisfies $f_M\left(p\right) = \langle q_\gamma \rangle_{\gamma \in \Gamma}$ as set mappings—see Lemma 9.2 below, then it follows that each such formally defined mapping in the context of Theorem 1.7 is a frame map in $Mpt\left(\bigotimes_{\gamma \in \Gamma} \tau_\gamma \right)$ and, in particular, preserves nonempty joins. Outside the context of Theorem 1.7, what properties should a map p defined formally by 9.1 have?

9.2 Lemma. The map $p : \bigotimes_{\gamma \in \Gamma} \tau_\gamma \rightarrow M$ formally defined using (9.1), namely by stipulating

$$p\left(u\right) = \bigvee_{\boxtimes_{i=1}^n u_{\gamma_i} \in \mathcal{B}_u} \left[\bigwedge_{i=1}^n q_{\gamma_i}\left(u_{\gamma_i}\right) \right],$$

has these properties:

1. The map p preserves $\underline{\perp}$ and $\underline{\top}$.
2. As set mappings $f_M\left(p\right) = \langle q_\gamma \rangle_{\gamma \in \Gamma}$.
3. The map p is isotone.
4. The map p preserves finite meets.

Altogether, the map $p : \bigotimes_{\gamma \in \Gamma} \tau_\gamma \rightarrow M$ is a $\mathbf{SLat}_{\perp}\left(\wedge\right)$ morphism such that $f_M\left(p\right) = \langle q_\gamma \rangle_{\gamma \in \Gamma}$ as set mappings.

Proof. *Ad (1).* Since each member of $\mathcal{B}_{\underline{\perp}}$ cannot be $\underline{\perp}$, it follows that $\mathrm{B}_{\underline{\perp}} = \varnothing$ and that

$$p\left(\underline{\perp}\right) = \bigvee_{\boxtimes_{i=1}^n u_{\gamma_i} \in \varnothing} \left[\bigwedge_{i=1}^n q_{\gamma_i}\left(u_{\gamma_i}\right) \right] = \perp;$$

and since $\mathcal{B}_{\underline{\top}}$ includes $\underline{\top}$ as, say, $\left(\pi_\beta\right)_L^{\leftarrow}\left(\underline{\top}_\beta\right)$, where $\underline{\top}_\beta$ is the whole space in τ_β, it follows

$$p\left(\underline{\top}\right) \geq q_\beta\left(\underline{\top}_\beta\right) = \top.$$

Ad (2). It suffices to show that $\forall \beta \in \Gamma,\ p_\beta = q_\beta$, where $p_\beta = p \circ (\pi_\beta)_L^{\leftarrow}$. Let $v_\beta \in \tau_\beta$. If $v_\beta = \perp_\beta$, then (1) and q_β being a frame map imply

$$p(v_\beta) = \perp = q_\beta(v_\beta);$$

and if $v_\beta \neq \perp_\beta$, then $(\pi_\beta)_L^{\leftarrow}(v_\beta) \neq \perp$, so that $(\pi_\beta)_L^{\leftarrow}(v_\beta) \in \mathcal{B}_{(\pi_\beta)_L^{\leftarrow}(v_\beta)}$ and

$$\begin{aligned} p_\beta(v_\beta) &= p\left((\pi_\beta)_L^{\leftarrow}(v_\beta)\right) \\ &= \bigvee_{\boxtimes_{i=1}^n u_{\gamma_i} \in \mathcal{B}_{(\pi_\beta)_L^{\leftarrow}(v_\beta)}} \left[\bigwedge_{i=1}^n q_{\gamma_i}(u_{\gamma_i})\right] \\ &= q_\beta(v_\beta), \end{aligned}$$

noting that the singleton cross product $\boxtimes_\beta v_\beta$ is precisely $(\pi_\beta)_L^{\leftarrow}(v_\beta)$.

Ad (3). If $u \leq v$, then $\mathcal{B}_u \subset \mathcal{B}_v$, and so $p(u) \leq p(v)$.

Ad (4). We first need two sublemmas.

9.2.1 Sublemma. Let $\boxtimes_{j=1}^m v_{\beta_j}$ be a non-bottom, basic L-open member of $\bigotimes_{\gamma\in\Gamma}\tau_\gamma$. Then

$$p\left(\boxtimes_{j=1}^m u_{\beta_j}\right) = \bigwedge_{j=1}^m q_{\beta_j}\left(v_{\beta_j}\right).$$

Proof of Sublemma. On one hand, $\boxtimes_{j=1}^m u_{\beta_j} \in \mathcal{B}_{\boxtimes_{j=1}^m u_{\beta_j}}$, so that

$$p\left(\boxtimes_{j=1}^m v_{\beta_j}\right) \geq \bigwedge_{j=1}^m q_{\beta_j}\left(v_{\beta_j}\right).$$

On the other hand, let $\boxtimes_{i=1}^n w_{\gamma_i} \in \mathcal{B}_{\boxtimes_{j=1}^m u_{\beta_j}}$. This means

$$\perp \neq \boxtimes_{i=1}^n w_{\gamma_i} \leq \boxtimes_{j=1}^m v_{\beta_j}$$

By adding a finite number of factors of $\top$ to either or both basic L-open sets, we may W.L.O.G. rewrite the above display as

$$\perp \neq \boxtimes_{k=1}^l w_{\delta_k} \leq \boxtimes_{k=1}^l v_{\delta_k}.$$

Product separation now says that $\forall k = 1, \ldots l$,

$$w_{\delta_k} \leq v_{\delta_k}, \quad q_{\delta_k}(w_{\delta_k}) \leq q_{\delta_k}(v_{\delta_k}),$$

so that

$$\bigwedge_{k=1}^l q_{\delta_k}(w_{\delta_k}) \leq \bigwedge_{k=1}^l q_{\delta_k}(v_{\delta_k}) = \bigwedge_{j=1}^m q_{\beta_j}\left(v_{\beta_j}\right).$$

It follows

$$p\left(\boxtimes_{j=1}^m v_{\beta_j}\right) \leq \bigwedge_{j=1}^m q_{\beta_j}\left(v_{\beta_j}\right),$$

concluding the proof of the sublemma. □

9.2.2 Sublemma. Let $\boxtimes_{j=1}^m u_{\beta_j}, \boxtimes_{j=1}^m v_{\beta_j}$ be non-bottom, basic L-open members of $\bigotimes_{\gamma\in\Gamma}\tau_\gamma$. Then

$$p\left(\left(\boxtimes_{j=1}^m u_{\beta_j}\right) \wedge \left(\boxtimes_{j=1}^m v_{\beta_j}\right)\right) = p\left(\boxtimes_{j=1}^m u_{\beta_j}\right) \wedge p\left(\boxtimes_{j=1}^m u_{\beta_j}\right).$$

Proof of Sublemma. This follows from 9.2.1, the associativity of $\wedge$, and the q_{β_j}'s preserving binary meets. □

Resumption of Ad (4) of 9.2. Let $u, v \in \bigotimes_{\gamma\in\Gamma}\tau_\gamma$. Since p is isotone by (3), we have that

$$p(u \wedge v) \leq p(u) \wedge p(v).$$

For the reverse direction, let $\boxtimes_{i=1}^{m} u_{\alpha_i} \in \mathcal{B}_u$, $\boxtimes_{j=1}^{n} v_{\beta_j} \in \mathcal{B}_v$; and as in the proof of 9.2.1, we W.L.O.G. rewrite these, respectively, as $\boxtimes_{k=1}^{l} u_{\delta_k}$, $\boxtimes_{k=1}^{l} v_{\delta_k}$. Now 9.2.2 says that

$$\begin{aligned} p\left(\boxtimes_{k=1}^{l} u_{\delta_k}\right) \wedge p\left(\boxtimes_{k=1}^{l} v_{\delta_k}\right) &= p\left(\left(\boxtimes_{k=1}^{l} u_{\delta_k}\right) \wedge \left(\boxtimes_{k=1}^{l} v_{\delta_k}\right)\right) \\ &= p\left(\boxtimes_{k=1}^{l}\left(u_{\delta_k} \wedge v_{\delta_k}\right)\right). \end{aligned}$$

If $\boxtimes_{k=1}^{l}\left(u_{\delta_k} \wedge v_{\delta_k}\right) = \perp$, then $p\left(\boxtimes_{k=1}^{l}\left(u_{\delta_k} \wedge v_{\delta_k}\right)\right) = \perp$ by (1), in which case

$$p\left(\boxtimes_{k=1}^{l} u_{\delta_k}\right) \wedge p\left(\boxtimes_{k=1}^{l} v_{\delta_k}\right) = p\left(\boxtimes_{k=1}^{l}\left(u_{\delta_k} \wedge v_{\delta_k}\right)\right) = \perp = p\left(u \wedge v\right);$$

and if $\boxtimes_{k=1}^{l}\left(u_{\delta_k} \wedge v_{\delta_k}\right) \neq \perp$, then $\boxtimes_{k=1}^{l}\left(u_{\delta_k} \wedge v_{\delta_k}\right) \in \mathcal{B}_{u \wedge v}$, in which case, using 9.2.1 and 9.2.2,

$$\begin{aligned} p\left(\boxtimes_{k=1}^{l} u_{\delta_k}\right) \wedge p\left(\boxtimes_{k=1}^{l} v_{\delta_k}\right) &= p\left(\boxtimes_{k=1}^{l}\left(u_{\delta_k} \wedge v_{\delta_k}\right)\right) \\ &= \bigwedge_{k=1}^{l} q_{\delta_k}\left(u_{\delta_k} \wedge v_{\delta_k}\right) \\ &\leq \bigvee_{\boxtimes_{i=1}^{n} w_{\gamma_i} \in \mathcal{B}_{u \wedge v}} \left[\bigwedge_{i=1}^{n} q_{\gamma_i}\left(w_{\gamma_i}\right)\right] \\ &= p\left(u \wedge v\right). \end{aligned}$$

Now invoking 9.2.1 yet again, we now have that

$$p\left(u \wedge v\right) = u.b.\left\{\bigwedge_{i=1}^{M} q_{\alpha_i}\left(u_{\alpha_i}\right) \wedge \bigwedge_{j=1}^{n} q_{\beta_i}\left(v_{\beta_i}\right) : \boxtimes_{i=1}^{m} u_{\alpha_i} \in \mathcal{B}_u,\ \boxtimes_{j=1}^{n} v_{\beta_j} \in \mathcal{B}_v\right\};$$

and hence, fixing $\boxtimes_{j=1}^{n} v_{\beta_j} \in \mathcal{B}_v$, that

$$p\left(u \wedge v\right) = u.b.\left\{\bigwedge_{i=1}^{M} q_{\alpha_i}\left(u_{\alpha_i}\right) \wedge \bigwedge_{j=1}^{n} q_{\beta_i}\left(v_{\beta_i}\right) : \boxtimes_{i=1}^{m} u_{\alpha_i} \in \mathcal{B}_u\right\},$$

and so by the frame law that

$$\begin{aligned} p\left(u\right) \wedge \bigwedge_{j=1}^{n} q_{\beta_i}\left(v_{\beta_i}\right) &= \left[\bigvee_{\boxtimes_{i=1}^{m} u_{\alpha_i} \in \mathcal{B}_u} \left(\bigwedge_{i=1}^{M} q_{\alpha_i}\left(u_{\alpha_i}\right)\right)\right] \wedge \bigwedge_{j=1}^{n} q_{\beta_i}\left(v_{\beta_i}\right) \\ &= \bigvee_{\boxtimes_{i=1}^{m} u_{\alpha_i} \in \mathcal{B}_u} \left[\bigwedge_{i=1}^{M} q_{\alpha_i}\left(u_{\alpha_i}\right) \wedge \bigwedge_{j=1}^{n} q_{\beta_i}\left(v_{\beta_i}\right)\right] \\ &\leq p\left(u \wedge v\right). \end{aligned}$$

Similarly, it now follows that

$$\begin{aligned} p\left(u\right) \wedge p\left(v\right) &= p\left(u\right) \wedge \left[\bigvee_{\boxtimes_{j=1}^{n} v_{\beta_j} \in \mathcal{B}_v} \bigwedge_{j=1}^{n} q_{\beta_i}\left(v_{\beta_i}\right)\right] \\ &= \bigvee_{\boxtimes_{j=1}^{n} v_{\beta_j} \in \mathcal{B}_v} \left[p\left(u\right) \wedge \bigwedge_{j=1}^{n} q_{\beta_i}\left(v_{\beta_i}\right)\right] \\ &\leq p\left(u \wedge v\right). \end{aligned}$$

This completes the proof that p preserves binary meets; and, invoking (1), we now have p preserves finite meets. $\square$

Motivated by 3.3 and the discussion above 9.2, let $\{(X_\gamma, \tau_\gamma) : \gamma \in \Gamma\}$ be a collection of L-topological spaces, where L is a frame, M is a subframe of L, and R is a binary relation on M given by

$$R \quad = \quad \left\{ \begin{array}{c} \left(\bigvee_{\alpha \in \mathcal{A}} p\left(a_\alpha\right),\ p\left(\bigvee_{\alpha \in \mathcal{A}} a_\alpha\right)\right) : \{a_\alpha\}_{\alpha \in \mathcal{A}} \subset \bigotimes_{\gamma \in \Gamma} \tau_\gamma, \\ \langle q_\gamma \rangle_{\gamma \in \Gamma} \in \prod_{\gamma \in \Gamma} Mpt\left(\tau_\gamma\right),\ p \in \mathbf{SLat}_\perp\left(\wedge\right)\left(\bigotimes_{\gamma \in \Gamma} \tau_\gamma, M\right), \\ f_M\left(p\right) = \langle q_\gamma \rangle_{\gamma \in \Gamma} \end{array} \right\}, \tag{9.3}$$

and let M/R be the quotient frame of M by R in the sense of 3.3 with nucleus $\nu_R : M \to M/R$.

9.4 Lemma. The following hold:

1. If M/R contains the bottom of M, then

$$(M/R)\,pt\left(\bigotimes_{\gamma\in\Gamma}\tau_\gamma\right) = \left\{\nu_R \circ q : q \in \mathbf{SLat}_\perp(\wedge)\left(\bigotimes_{\gamma\in\Gamma}\tau_\gamma, M\right)\right\}.$$

2. If M/R is a subframe of M, then $\forall A \in |\mathbf{Loc}|$, then

$$(M/R)\,pt\,(A) = \{\nu_R \circ q : q \in Mpt\,(A)\}\,.$$

Proof. *Ad (1).* For "$\subset$", let $r \in (M/R)\,pt\left(\bigotimes_{\gamma\in\Gamma}\tau_\gamma\right)$. Then M/R containing the bottom of M yields that $\hookrightarrow : M/R \to M$ is a morphism in $\mathbf{SLat}_\perp(\wedge)$; and so r may be viewed as a member of $\mathbf{SLat}_\perp(\wedge)\left(\bigotimes_{\gamma\in\Gamma}\tau_\gamma, M\right)$. Further, since $\nu_R : M \to M/R$ is the identity on M/R, then

$$\nu_R \circ r = r.$$

It follows that $r \in \left\{\nu_R \circ q : q \in \mathbf{SLat}_\perp(\wedge)\left(\bigotimes_{\gamma\in\Gamma}\tau_\gamma, M\right)\right\}$. Now let

$$r \in \mathbf{SLat}_\perp(\wedge)\left(\bigotimes_{\gamma\in\Gamma}\tau_\gamma, M\right).$$

Then $\nu_R \circ r$ preserves all finite meets and the empty join since r does and ν_R is a frame map. Now let $\{a_\alpha\}_{\alpha\in\mathcal{A}} \subset \bigotimes_{\gamma\in\Gamma}\tau_\gamma$. Then from 3.3.1(2) and the fact ν_R is a frame map, it follows:

$$\nu_R\left(r\left(\bigvee_{\alpha\in\mathcal{A}} a_\alpha\right)\right) = \nu_R\left(\bigvee_{\alpha\in\mathcal{A}} r\,(a_\alpha)\right) = \bigvee_{\alpha\in\mathcal{A}} \nu_R\,(r\,(a_\alpha))\,.$$

Now $\nu_R \circ r$ is a frame map from $\bigotimes_{\gamma\in\Gamma}\tau_\gamma$ to M/R, hence is in $(M/R)\,pt\left(\bigotimes_{\gamma\in\Gamma}\tau_\gamma\right)$, and so "$\supset$" holds.

Ad (2). Let $A \in |\mathbf{Loc}|$. If $r \in (M/R)\,pt\left(\bigotimes_{\gamma\in\Gamma}\tau_\gamma\right)$, then r is a frame map into M—M/R is a subframe of M, and $\nu_R \circ r = r$; so that $r \in Mpt\,(A)$. And the reverse inclusion follows since $\nu_R \circ r$ is a composition of frame maps. □

9.5 Definition (join separation). Given a subframe M of frame L, a collection $\{(X_\gamma, \tau_\gamma) : \gamma \in \Gamma\}$ of L-topological spaces is *M-join-separated* if M/R is a subframe of M, and L is (M/R)-spatial, where R is defined as in 9.3 above; and this collection of L-topological spaces is *join-separated* if M-join separation holds with $M = L$.

9.6 Proof of Theorem 1.8. Choosing $M = L\,/\,R$ in 6.11 above, it suffices to show that

$$f_M : \left(Mpt\left(\bigotimes_{\gamma\in\Gamma}\tau_\gamma\right), (\Phi_M)^{\to}\left(\bigotimes_{\gamma\in\Gamma}\tau_\gamma\right)\right) \to \left(\prod_{\gamma\in\Gamma} Mpt\,(\tau_\gamma)\,, \bigotimes_{\gamma\in\Gamma}(\Phi_M)^{\to}(\tau_\gamma)\right)$$

is surjective with respect to *frame* mappings; and choosing $M = L$ in 9.2 above, we therefore have that

$$f_L : \left(Lpt\left(\bigotimes_{\gamma\in\Gamma}\tau_\gamma\right), (\Phi_L)^{\to}\left(\bigotimes_{\gamma\in\Gamma}\tau_\gamma\right)\right) \to \left(\prod_{\gamma\in\Gamma} Lpt\,(\tau_\gamma)\,, \bigotimes_{\gamma\in\Gamma}(\Phi_L)^{\to}(\tau_\gamma)\right)$$

is surjective with respect to $\mathbf{SLat}_\perp(\wedge)$ morphisms: more precisely, given a tuple $\langle q_\gamma\rangle_{\gamma\in\Gamma} \in \prod_{\gamma\in\Gamma} Lpt\,(\tau_\gamma)$, there is a $\mathbf{SLat}_\perp(\wedge)$ morphism p such that as $\mathbf{SLat}_\perp(\wedge)$ morphisms, $f_L\,(p) = \langle q_\gamma\rangle_{\gamma\in\Gamma}$. Now suppose we start with $\langle q_\gamma\rangle_{\gamma\in\Gamma} \in \prod_{\gamma\in\Gamma} Mpt\,(\tau_\gamma)$. Then since M is a subframe of L, it follows from 9.4(2) above that

$\langle q_\gamma \rangle_{\gamma\in\Gamma} \in \prod_{\gamma\in\Gamma} Lpt\,(\tau_\gamma)$. Then there is a $\mathbf{SLat}_\perp\,(\wedge)$ morphism $\widehat{p}$ such that as set mappings, $f_L\,(\widehat{p}) = \langle q_\gamma \rangle_{\gamma\in\Gamma}$. Now using the nucleus $\nu_R : L \to M/R \hookrightarrow M$, we have from 9.4(1) above that $p \equiv \nu_R \circ \widehat{p} \in Mpt\left(\bigotimes_{\gamma\in\Gamma} \tau_\gamma\right)$. It follows that $f_M\,(p) = \langle p_\gamma \rangle_{\gamma\in\Gamma}$ by definition of f_M; and further, $\forall\gamma \in \Gamma$, we have that

$$\begin{aligned} p_\gamma &= p \circ (\pi_\gamma)_L^{\leftarrow} \\ &= (\nu_R \circ \widehat{p}) \circ (\pi_\gamma)_L^{\leftarrow} \\ &= \nu_R \circ \left(\widehat{p} \circ (\pi_\gamma)_L^{\leftarrow}\right) \\ &= \nu_R \circ \widehat{p}_\gamma \quad \text{(definition of } \widehat{p}_\gamma\text{)} \\ &= \nu_R \circ q_\gamma \quad (f_L\,(\widehat{p}) = \langle q_\gamma \rangle_{\gamma\in\Gamma} \text{ as set mappings)} \\ &= q_\gamma \quad \text{(proof of 9.4).} \end{aligned}$$

Hence $f_M\,(p) = \langle q_\gamma \rangle_{\gamma\in\Gamma}$, f_M is now surjective with respect to frame mappings, and so f_M is an L-homeomorphism and Theorem 1.8 now follows from 6.11. □

9.7 Proof of Theorem 1.9. Instantiate 6.11 by choosing $M = L$. Then the map

$$f_L : \left(Lpt\left(\bigotimes_{\gamma\in\Gamma} \tau_\gamma\right), (\Phi_L)^{\to}\left(\bigotimes_{\gamma\in\Gamma} \tau_\gamma\right)\right) \to \left(\prod_{\gamma\in\Gamma} Lpt\,(\tau_\gamma)\,, \bigotimes_{\gamma\in\Gamma} (\Phi_L)^{\to}\,(\tau_\gamma)\right)$$

is an L-embedding by 2.4.1(3). Apply 2.4.2(4) to finish the proof. □

References

[1] J. T. Denniston, S. E. Rodabaugh, *Functorial relationships between lattice-valued topology and topological systems*, Quaestiones Mathematicae **32:2**(2009), 139–186.

[2] D. H. Dowker, D. Papert, *Sums in the category of frames*, Houston J. Math. **3**(1976), 7–15.

[3] J. L. Frith, *Structured frames*, Ph.D. Dissertation (University of Cape Town, Cape Town, South Africa, 1987).

[4] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, D. S. Scott, *A Compendium of Continuous Lattices*, Springer Verlag (Berlin, Heidelberg, New York), 1980.

[5] U. Höhle, *Fuzzy topologies and the topological space objects in a topos*, Fuzzy Sets and Systems **19**(1986), 299–304.

[6] U. Höhle, *Locales and L-topologies*, Mathematik-Arbeitpapiere (Universität Bremen) **48**(1997), 223–250.

[7] J. R. Isbell, *Atomless parts of spaces*, Math. Scand.**31**(1972), 5–32.

[8] J.R. Isbell, *Product spaces in locales*, Proc. Amer. Math. Soc. **81**(1981), 116-118.

[9] P. T. Johnstone, *Stone Spaces*, Cambridge University Press (Cambridge), 1982.

[10] W. Kotzé, *Lattice morphisms, sobriety, and Urysohn lemmas*, pp. 257–274 in S. E. Rodabaugh, E. P. Klement, U. Höhle, eds, *Application Of Category Theory To Fuzzy Subsets*, Theory and Decision Library—Series B: Mathematical and Statistical Methods **14**(1992), Kluwer Academic Publishers (Boston/Dordrecht/London).

[11] W. Kotzé, *Sobriety and semi-sobriety of* L*-topological spaces*, Quaestiones Mathematicae **24**(2001), 549–554.

[12] T. Plewe, *Localic products of spaces*, Proceedings of the London Mathematical Society, **73:3**(1996), 642–678.

[13] A. Pultr, *Frames*, pp. 789–858 in M. Hasewinkel, ed., Handbook of Algebra **3**(2003), North-Holland Elsevier (Amsterdam)

[14] A. Pultr, S. E. Rodabaugh, *Lattice-valued frames, functor categories, and classes of sober spaces*, pp. 153–187 in S. E. Rodabaugh, E. P. Klement, eds, *Topological And Algebraic Structures In Fuzzy Sets: A Handbook of Recent Developments in the Mathematics of Fuzzy Sets*, Trends in Logic **20**(2003), Kluwer Academic Publishers (Boston/Dordrecht/London).

[15] A. Pultr, S. E. Rodabaugh, *Examples for different sobrieties in fixed-basis topology*, pp. 427–440 in S. E. Rodabaugh, E. P. Klement, eds, *Topological And Algebraic Structures In Fuzzy Sets: A Handbook of Recent Developments in the Mathematics of Fuzzy Sets*, Trends in Logic **20**(2003), Kluwer Academic Publishers (Boston/Dordrecht/London).

[16] A. Pultr, S. E. Rodabaugh, *Category theoretic aspects of chain-valued frames: Part I: Categorical and presheaf theoretic foundations*, Fuzzy Sets and Systems **159:5**(2008), 501–528.

[17] A. Pultr, S. E. Rodabaugh, *Categorical aspects of chain-valued frames: Part II: Applications to lattice-valued topology*, Fuzzy Sets and Systems **159:5**(2008), 529–558.

[18] S. E. Rodabaugh, *A point set lattice-theoretic framework* $\mathbb{T}$ *which contains* **Loc** *as a subcategory of singleton spaces and in which there are general classes of Stone representation and compactification theorems*, preprint, first draft February 1986 / second draft April 1987, Youngstown State University Central Printing Office (Youngstown, Ohio).

[19] S. E. Rodabaugh, *Point-set lattice-theoretic topology*, Fuzzy Sets and Systems **40**(1991), 297–345.

[20] S. E. Rodabaugh, *Categorical frameworks for Stone representation theorems*, pp. 178–231 in S. E. Rodabaugh, E. P. Klement, U. Höhle, eds, *Application Of Category Theory To Fuzzy Subsets*, Theory and Decision Library—Series B: Mathematical and Statistical Methods **14**(1992), Kluwer Academic Publishers (Boston/Dordrecht/London).

[21] S. E. Rodabaugh, ed., *Roundtable: Model-theoretic* vis-à-vis *non-model-theoretic fuzzy set theories*, pp. 337–345 in S. E. Rodabaugh, E. P. Klement, U. Höhle, eds, *Application Of Category Theory To Fuzzy Subsets*, Theory and Decision Library—Series B: Mathematical and Statistical Methods **14**(1992), Kluwer Academic Publishers (Boston/Dordrecht/London).

[22] S. E. Rodabaugh, *Applications of localic separation axioms, compactness axioms, representations, and compactifications to poslat topological spaces*, Fuzzy Sets and Systems **73**(1995), 55–87.

[23] S. E. Rodabaugh, *Categorical foundations of variable-basis fuzzy topology*, pp. 273–388 in U. Höhle, S. E. Rodabaugh, eds, *Mathematics Of Fuzzy Sets: Logic, Topology, And Measure Theory, The Handbooks of Fuzzy Sets Series*, Volume **3**(1999), Kluwer Academic Publishers (Boston/Dordrecht/London).

[24] S. E. Rodabaugh, *Separation axioms: representation theorems, compactness, and compactifications*, pp. 481–552, in U. Höhle, S. E. Rodabaugh, eds, *Mathematics Of Fuzzy Sets: Logic, Topology, And Measure Theory, The Handbooks of Fuzzy Sets Series*, Volume **3**(1999), Kluwer Academic Publishers (Boston/Dordrecht/London).

[25] S. E. Rodabaugh, *Relationship of algebraic theories to powerset theories and fuzzy topological theories for lattice-valued mathematics*, International Journal of Mathematics and Mathematical Sciences Vol. **2007(3)**, Article ID 43645, 71 pages, **2007**. doi:10.1155/2007/43645. http://www.hindawi.com/getarticle.aspx?doi=10.1155/2007/43645.

[26] S. E. Rodabaugh, *Necessity of non-stratified and anti-stratified spaces in lattice-valued topology*, Fuzzy Sets and Systems, **161**(2010), 1253–1269.

[27] S. A. Solovyov, *Variable-basis topological systems versus variable-basis topological spaces*, Soft Computing, to appear.

[28] S. J. Vickers, *Topology Via Logic,* Cambridge Tracts in Theoretical Computer Science **5**(1989), Cambridge University Press (Cambridge).